Math 670: Day 1

Consider the space of k-dimensional linear subspaces in $\mathbb{R}^n$ (or $\mathbb{C}^n$). This space is called the Grassmannian of k-planes in $\mathbb{R}^n$ & denoted $G_k(\mathbb{R}^n)$ (or $Gr(k,n)$ or $Gr(n,k)$ or ...).

This space pops up all over mathematics:

- Algebraic Geometry ⇝ ex. of a projective variety (via the Plücker embedding) & more specifically as a flag variety.
  First considered here b/c they arise in counting problems, so...
- Combinatorics ⇝ esp. Schubert calculus, developed to count # of points of intersection of lines, e.g. "how many lines in 3-space intersect 4 given lines?"
- Differential geometry ⇝ these are classic examples of so-called homogeneous spaces, which can be viewed as quotients of Lie groups.
- Applied math ⇝
  - Signal detection (e.g. Kirby, Peterson, et al)
  - Blind source separation
  - optimization (linear & quadratic constraints put you on a Grassmannian... e.g., symmetric eigenvalue problem)
  - Closed Random Walks (my personal favorite... closure constraint exactly corresponds to being on the Grassmannian)
  - etc....

Ex: $G_1(\mathbb{R}^2)$ is the space of lines through the origin in the plane... clearly, this is just $\mathbb{RP}^1 \cong S^1$.

- In turn, $G_1(\mathbb{R}^3)$ is simply $\mathbb{RP}^2$... in general, $G_1(\mathbb{R}^n) = \mathbb{RP}^{n-1}$.
- Now, $G_2(\mathbb{R}^3)$ is the space of planes through the origin in $\mathbb{R}^3$... which is equivalent to $G_1(\mathbb{R}^3) = \mathbb{RP}^2$ by mapping a plane to its orthogonal complement.

Also, $G_2(\mathbb{R}^3)$ can be interpreted as the set of all lines (not just through the origin) in projective space $\mathbb{RP}^2$...

Complete $\mathbb{R}^2$ to $\mathbb{RP}^2$ by embedding $\mathbb{R}^2 \to \mathbb{R}^3 - \{\vec{0}\}$ as the set $\{(1,x,y)\} \cup \{(0,x,y) : x \neq 0 \text{ or } y \neq 0\}$

and identifying points related by scalar multiplication.

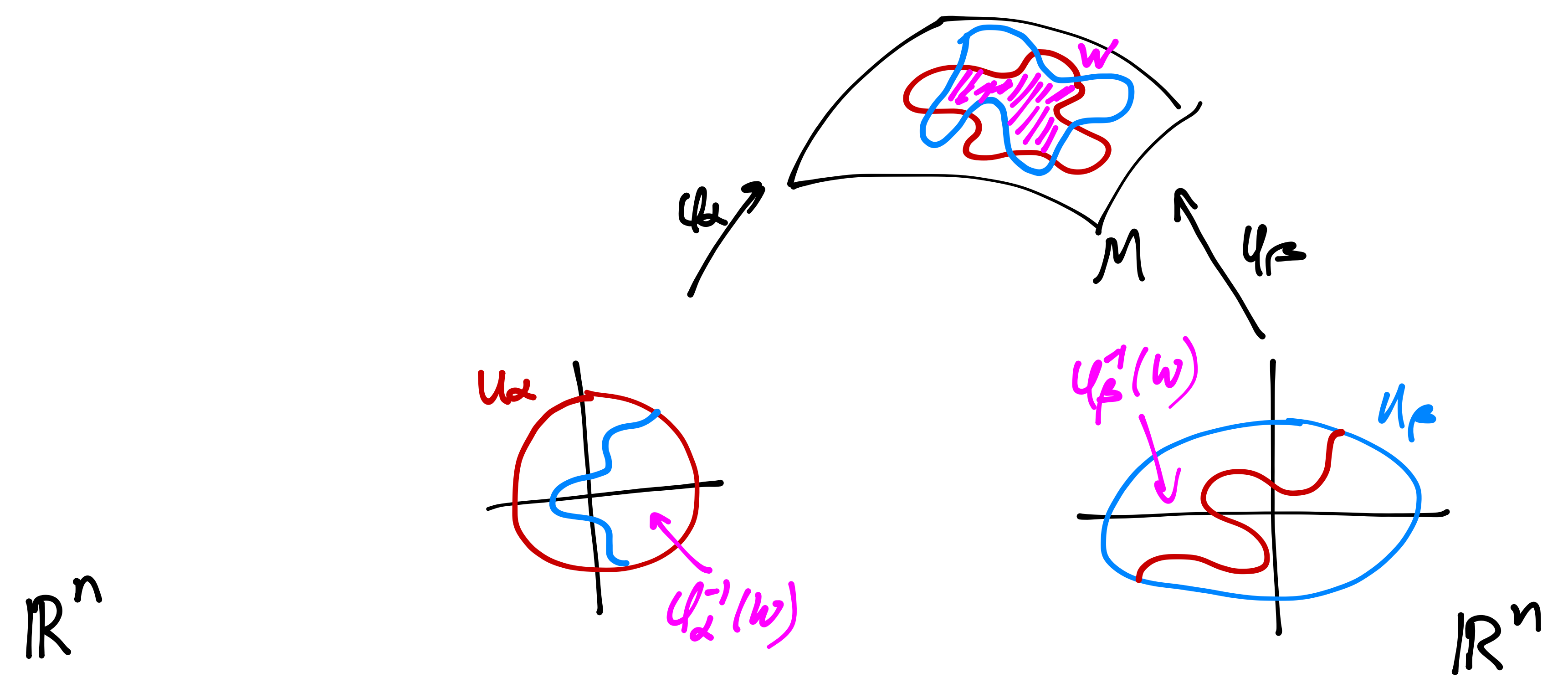
Then a plane thru the origin in $\mathbb{R}^3$ determines a (unique) line in this space.

- $G_2(\mathbb{R}^4)$ turns out to be isometric to $S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$
- We can play the same game with $\mathbb{C}^n$... let $G_k(\mathbb{C}^n)$ be the set of k-dimensional (complex) linear subspaces of $\mathbb{C}^n$. So for example $G_1(\mathbb{C}^2)$ is the space of complex lines thru the origin in $\mathbb{C}^2$, better known as $\mathbb{CP}^1$ (or $\mathbb{P}^1$ if you're an algebraic geometer). As some of you probably know, this is just a copy of S^2 (of radius $1/2$ if you're being careful)... one way to see this is by way of the Hopf fibration, which is one of the most beautiful (& important!) constructions of the last hundred years.

The goal is to build up a (differential) geometric understanding of the Grassmannians & other similar spaces... to get there we'll need to build up some of the theory of differentiable manifolds, differential forms, and Lie groups.

Def: A differentiable manifold of dimension n is a Hausdorff, second countable topological space M together w/ a family of injective maps $\varphi_\alpha : U_\alpha \subseteq \mathbb{R}^n \to M$ of open sets $U_\alpha \subseteq \mathbb{R}^n$ s.t.:

1. $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$
2. For any α, β s.t. $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) = W \neq \emptyset$, the sets $\varphi_\alpha^{-1}(W)$ & $\varphi_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ & the maps $\varphi_\beta^{-1} \circ \varphi_\alpha$ & $\varphi_\alpha^{-1} \circ \varphi_\beta$ are smooth (C^∞).
3. The family $\{(U_\alpha, \varphi_\alpha)\}$ is maximal w.r.t. 1 & 2.



Ex: · The maximal family containing $(\mathbb{R}^n, id)$ obviously makes $\mathbb{R}^n$ into an n-dim'l manifold.