Math 670 HW #6

Due 11:00 AM Friday, May 8

1. We can define a map $\varphi : S^3 \to \mathrm{SO}(3)$ as follows: given a point $q \in S^3$, which we'll think of as a unit quaternion, the rotation $\varphi(q) \in \mathrm{SO}(3)$ can be specified by describing what it does to a point $v \in S^2$. Points in S^2 can be thought of as unit purely imaginary quaternions (i.e., unit quaternions of the form $v = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$), and then

$$\varphi(q)(v) := qv\overline{q}.$$

   Prove that φ is a Lie group homomorphism, identify the kernel, and use this to conclude that $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{RP}^3$.

2. Give a completely different proof that $\mathrm{SO}(3) \cong \mathbb{RP}^3$.

3. Prove that the map $F : S^3 \times S^3 \to \mathrm{SO}(4)$ given by

$$F(p, q) = v \mapsto pv\overline{q}$$

   is a surjective Lie group homomorphism with kernel $\{\pm(1, 1)\}$. Conclude that $\mathrm{SO}(4)$ is diffeomorphic to $(S^3 \times S^3)/\{(x, y) \sim -(x, y)\}$ (which is diffeomorphic but not Lie group isomorphic to $S^3 \times \mathbb{RP}^3$).

4. (Challenge Problem) Recall that the space of planar triangles up to translation, rotation, and scaling can be identified with $G_2\mathbb{R}^3 \cong G_1\mathbb{R}^3 = \mathbb{RP}^2$.

   (a) What is the subset of *acute* triangles, thought of either as a subset of $\mathbb{RP}^2$ or (after lifting to the double cover) as a subset of S^2. There are, of course, various ways of specifying this subset: for example, you could give a set of inequalities satisfied precisely by the points in the region, you could specify the boundary as a set of parametrized curves, etc. The region on the sphere is shown below, so you can check your answer.

   (b) What is the area (or measure, if you prefer that terminology) of the subset of acute triangles? (*Hint*: Turn this into a multivariable calculus problem.)