Math 670 HW #5

Due 11:00 AM Wednesday, April 22

1. Give an example of a Lie group G and a Lie subgroup H which is not closed in G (in particular, this means that G/H is not necessarily a manifold, or even Hausdorff).

2. Let G be a Lie group.

   (a) Show that the set of right-invariant vector fields on G forms a Lie algebra with bracket given by the Lie bracket of vector fields. Note that the right-invariant vector fields form a vector space which is isomorphic to T_eG.

   (b) Let $\mathrm{inv} : G \to G$ be given by $\mathrm{inv}(g) = g^{-1}$. Prove that if X is a left-invariant vector field on G, then $d\,\mathrm{inv}(X)$ is a right-invariant vector field whose value at e is $-X_e$.

   (c) Prove that the map $d\,\mathrm{inv}$ from left-invariant vector fields to right-invariant vector fields is a Lie algebra isomorphism. (The point is that we could just have well chosen to interpret the Lie algebra of G as the right-invariant vector fields rather than the left-invariant ones.)

3. Let exp denote the matrix exponential map, meaning that, for $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$,

$$\exp(A) = e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

   Remember that $\mathfrak{so}(n)$ can be identified with the set of $n \times n$ skew-symmetric matrices.

   (a) Prove that $\exp : \mathfrak{so}(n) \to SO(n)$ is well-defined and surjective. (*Hint:* for surjectivity, think about what the real Jordan normal form of a skew-symmetric matrix A and of an orthogonal matrix B look like.)

   (b) Suppose $A, B \in \mathfrak{so}(n)$ so that $[A, B] = 0$. Show that

$$\exp(A + B) = \exp A \exp B.$$

4. Prove that $U(n)$ and $U(1) \times SU(n)$ are diffeomorphic but not isomorphic as Lie groups.