Math 670 HW #4

Due 11:00 AM Monday, March 23

1. Consider the *Heisenberg group*

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Use the techniques demonstrated in class for the group of affine transformations of the line to do the following:

(a) Find a basis for the left-invariant vector fields on H, and compute their Lie brackets.

(b) Express the left-invariant vector fields in terms of the (global) coordinates x, y, and z (i.e., you should be able to write a vector field V at a point $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in H$ as $a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$, where a, b, and c are functions of (x, y, z)).

(c) Let $\varphi_1, \varphi_2, \varphi_3$ be the dual basis of left-invariant 1-forms on H and compute their exterior derivatives using the equation $d\varphi(X, Y) = X(\varphi(Y)) - Y(\varphi(X)) - \varphi([X, Y])$, which was a corollary of Cartan's magic formula.

(d) Express the φ_i in x, y, z coordinates.

(e) Use the Maurer-Cartan equation $d\Omega + \Omega \wedge \Omega = 0$ to recompute the exterior derivatives of these 1-forms.

(f) Use the adjoint action of H on $\mathfrak{h}$ to determine all of the bi-invariant 1-, 2-, and 3-forms on H (recall that for each $h \in H$, the map $\mathrm{Ad}_h : \mathfrak{h} \to \mathfrak{h}$ is the differential at the identity of conjugation by h). Verify that these bi-invariant forms are all closed.

2. Consider the special orthogonal group $SO(3)$ of all 3×3 matrices B such that

$$BB^\top = I \qquad \text{and} \qquad \det B = 1.$$

The one-parameter subgroup $\left\{ \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$ of $SO(3)$ has tangent vector $E_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ at the identity; likewise, we have tangent vectors E_{ij} at the identity with $1 \leq i \neq j \leq 3$ and $E_{ji} = -E_{ij}$. We get a basis for $T_I SO(3)$ given by $\{E_{12}, E_{13}, E_{23}\}$.

(a) We can identify $E_{ij} \in T_I SO(3)$ with the left-invariant vector field $V_{ij} \in \mathfrak{so}(3)$. Show that $[V_{ij}, V_{k\ell}] = -\delta_{jk} V_{i\ell}$ (where by convention $V_{ii} = 0$ for any i).

(b) Let φ_{ij} be the dual basis of left-invariant 1-forms and compute their exterior derivatives. You can do this either using Cartan's magic formula or by showing that

$$\Omega = \begin{pmatrix} 0 & -\varphi_{12} & -\varphi_{13} \\ \varphi_{12} & 0 & -\varphi_{23} \\ \varphi_{13} & \varphi_{23} & 0 \end{pmatrix}$$

and using the Maurer-Cartan equation.

(c) Find all the bi-invariant forms on $SO(3)$ and use this to compute the de Rham cohomology groups of $SO(3)$.

3. Let G be a compact Lie group and assume $\langle \cdot, \cdot \rangle$ is an Ad-invariant inner product on $\mathfrak{g}$ (an Ad-invariant inner product on $\mathfrak{g}$ is one that satisfies $\langle X, Y \rangle = \langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y \rangle$ for all $g \in G$ and for any $X, Y \in \mathfrak{g}$).

   Define $\tau_e : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by
$$\tau(X, Y, Z) = \langle [X, Y], Z \rangle.$$

   (a) Show that τ_e is alternating. Since τ_e is clearly multilinear, this means τ_e gives an element of $\bigwedge^3(\mathfrak{g}^*)$.

   (b) Extend τ_e to a left-invariant 3-form on G in the usual way: for each $g \in G$, define $\tau_g := L^*_{g^{-1}} \tau_e$. Prove that $\tau \in \Omega^3(G)$ is bi-invariant. The bi-invariant 3-form τ is called the *fundamental 3-form* of the Lie group G.

   (c) Explicitly compute the fundamental 3-form of $SO(3)$ in terms of the φ_{ij} from the previous problem.

Challenge Problems

- Prove that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.
- Prove that $SO(4)$ is homeomorphic to $S^3 \times \mathbb{R}P^3$ (though they are *not* isomorphic as groups).
- Re-do problems 2 and 3(c) for $SO(4)$.