Math 670 HW #3

Due 11:00 AM Friday, March 6

1. Let V be an n-dimensional inner product space and, as in Problem 5 from HW 2, extend the inner product to the exterior powers of V. Let $\gamma : \Lambda^{k+1}(V) \to \Lambda^k(V)$ be the adjoint of left exterior multiplication by $v \in V$, meaning that

$$\langle \gamma(\omega), \eta \rangle = \langle \omega, v \wedge \eta \rangle$$

for any $\omega \in \Lambda^{k+1}(V)$ and $\eta \in \Lambda^k(V)$. Prove that

$$\gamma(\omega) = (-1)^{nk} \star (v \wedge (\star\omega)).$$

2. (**Cartan Lemma**) Let $k \leq n$ and let $\omega_1 \dots, \omega_k \in \Omega^1(M^n)$ so that the ω_i are linearly independent pointwise. Let $\theta_1, \dots, \theta_k \in \Omega^1(M)$ so that

$$\sum_{i=1}^{k} \theta_i \wedge \omega_i = 0.$$

Prove that there exist smooth functions $A_{ij} \in C^\infty(M)$ with $A_{ij} = A_{ji}$ so that

$$\theta_i = \sum_{j=1}^{k} A_{ij}\omega_j \quad \text{for all } i \in \{1, \dots, k\}.$$

(Note that this is trivially true when $k = n$ and only interesting [and surprising!] when $k < n$.)

3. Let $\omega \in \Omega^1(\mathbb{R}^3)$ be given by

$$z\,\mathrm{d}y + xy\,\mathrm{d}z$$

and let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x, y) = (x^2, x, xy)$. We know from class that $d(f^*\omega) = f^*d\omega$, but show this explicitly by computing both sides.

4. A differential form $\omega \in \Omega^k(M)$ is called *closed* if it is in the kernel of the exterior derivative, meaning that $d\omega = 0$. Such a form is called *exact* if it is in the image of the exterior derivative, meaning that $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$.

Show that the 1-form ω on $\mathbb{R}^2 - \{0\}$ given by

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

is closed but not exact. (Equivalently, the vector field $\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ is curl-free but is not the gradient of any function.)

5. Let M^n be a closed manifold (i.e., a compact manifold without boundary) and let $\omega \in \Omega^1(M)$ so that $\omega_p \neq 0$ for all $p \in M$. Show that ω is not exact.