Math 670 HW #2

Due 11:00 AM Friday, February 20

1. (a) Show that, for $v \in \Lambda^k(V)$ and $w \in \Lambda^\ell(V)$,

$$w \wedge v = (-1)^{k\ell} v \wedge w.$$

   (b) Prove the universal property of the exterior product (feel free to assume the universal property of the tensor product).

2. Let $A : V \to W$ be a linear map between vector spaces.

   (a) Show that the induced map $\Lambda^k(V) \to \Lambda^k(W)$ is well-defined by

$$v_1 \wedge \ldots \wedge v_k \mapsto Av_1 \wedge \ldots \wedge Av_k$$

   (extending linearly to sums).

   (b) Show that the transpose $A^* : W^* \to V^*$ defines a map $\Lambda^k(W^*) \to \Lambda^k(V^*)$.

   (c) Show that, if $V = W$ is an n-dimensional vector space, then the map $\Lambda^n(V) \to \Lambda^n(V)$ is multiplication by $\det A$.

3. Show that the vectors $v_1, \ldots, v_k \in V$ are linearly independent if and only if $v_1 \wedge \ldots \wedge v_k \neq 0$ as an element of $\Lambda^k(V)$.

4. We say that an element of $\Lambda^k(V)$ is *decomposable* if it can be written as $v_1 \wedge \ldots \wedge v_k$.

   (a) Suppose $v, w, x, y \in V$. Find necessary and sufficient conditions for $v \wedge w + x \wedge y \in \Lambda^2(V)$ to be decomposable.

   (b) Show that $\omega \in \Lambda^2(\mathbb{R}^4)$ is decomposable if and only if $\omega \wedge \omega = 0$.

5. Let V be an n-dimensional inner product space. We can extend the inner product from V to all of $\Lambda(V)$ by setting the inner product of homogeneous elements of different degrees equal to zero and by letting

$$\langle w_1 \wedge \ldots \wedge w_k, v_1 \wedge \ldots \wedge v_k \rangle = \det \left(\langle w_i, v_j \rangle \right)_{i,j}$$

   and extending bilinearly.

   Since $\Lambda^n(V)$ is a one-dimensional real vector space, $\Lambda^n(V) - \{0\}$ has two components. An *orientation* on V is a choice of component of $\Lambda^n(V) - \{0\}$. If V is an oriented inner product space, then there is a linear map $\star : \Lambda(V) \to \Lambda(V)$ called the star map, which is defined by requiring that for any orthonormal basis $e_1, \ldots, e_n$ for V,

$$\star(1) = \pm e_1 \wedge \ldots \wedge e_n, \qquad \star\left(e_1 \wedge \ldots \wedge e_n\right) = \pm 1,$$
$$\star(e_1 \wedge \ldots \wedge e_k) = \pm e_{k+1} \wedge \ldots \wedge e_n,$$

   where in each case we take "+" if $e_1 \wedge \ldots \wedge e_n$ is in the preferred component of $\Lambda^n(V)$ and we take "−" otherwise. Notice that $\star : \Lambda^k(V) \to \Lambda^{n-k}(V)$.

(a) Prove that if $e_1, \ldots, e_n$ is an orthonormal basis for V, then the $e_{i_1} \wedge \ldots \wedge e_{i_k}$ with $1 \leq i_1 < \ldots < i_k \leq n$ and $1 \leq k \leq n$ give an orthonormal basis for $\Lambda(V)$.

(b) Prove that, as a map $\Lambda^k(V) \to \Lambda^{n-k}(V)$, $\star\star = (-1)^{k(n-k)}$.

(c) Prove that, for $\omega, \eta \in \Lambda^k(V)$, their inner product is given by

$$\langle \omega, \eta \rangle = \star(\omega \wedge \star\eta) = \star(\eta \wedge \star\omega).$$