Math 670 HW #1

Due 11:00 AM Friday, February 6

1. A smooth manifold M is called *orientable* if there exists a collection of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ so that, for every α, β such that $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \emptyset$, the differential of the change of coordinates $\phi_\beta^{-1} \circ \phi_\alpha$ has positive determinant.

   (a) Show that for any n, the sphere S^n is orientable.

   (b) Prove that, if M and N are smooth manifolds and $f : M \to N$ is a local diffeomorphism at all points of M, then N being orientable implies that M is orientable. Is the converse true?

2. Supply the details for the proof that, if $f : GL(n, \mathbb{R}) \to \{n \times n \text{ symmetric matrices}\}$ is given by $f(A) = AA^\top$, then
$$O(n) = f^{-1}(I_n)$$
is a submanifold of $GL(n, \mathbb{R})$ of dimension $\frac{n(n-1)}{2}$. (Hint: it may be helpful to remember that a symmetric matrix C can always be written as $C = \frac{1}{2}C + \frac{1}{2}C^\top$.)

3. Prove that, if X, Y, and Z are smooth vector fields on a smooth manifold M and $a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$, then

   (a) $[X, Y] = -[Y, X]$ (anticommutivity)

   (b) $[aX + bY, Z] = a[X, Z] + b[X, Z]$ (linearity)

   (c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity)

   (d) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.

4. A *Riemannian manifold* is a pair (M, g) where is a M is a smooth manifold and g is a *Riemannian metric*, which is a smooth choice of inner product on each tangent space of M. More precisely, at each $p \in M$, the metric is a positive definite inner product g_p on T_pM, meaning that $g_p : T_pM \times T_pM \to \mathbb{R}$ satisfies the inner product axioms. Moreover, this choice smoothly depends on p, meaning that, if X and Y are vector fields on M, then $g(X, Y)$ is a C^∞ function on M.

   Prove that there exists a Riemannian metric on every smooth manifold. (Hint: use a partition of unity argument. For a refresher on partitions of unity, see, e.g., Definition 1.8 and Theorem 1.11 from Warner, Theorem 36.1 and the preceding definition in Munkres, etc.)