Math 617: Days 7-8

The Hahn-Kolmogorov Thm gives us the necessary infrastructure to build Lebesgue measure. In fact, Lebesgue measure is a special case of a **Lebesgue-Stieltjes measure**, which is a general class of measures defined on the Borel sets of $\mathbb{R}$.

The idea of Lebesgue-Stieltjes is the following: Imagine you have some increasing function F on $\mathbb{R}$. We want to say that the measure of an interval $[a,b] = F(b) - F(a)$. The classic example is when F is a **cumulative distribution function** (cdf) for some probability distribution on $\mathbb{R}$: a cdf is defined by $F(x) = P(X \leq x)$ where X is a random variate chosen from the probability dist.

For example, the cdf of the standard Gaussian is $F(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt = \frac{1}{2} + \int_0^x \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$, where erf is the error function.

On the other hand, the cdf of the δ distribution is the Heaviside function $H(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x \geq 0 \end{cases}$

This suggests we want to allow discontinuous functions...

Thm (Existence of Lebesgue-Stieltjes measure) Let $F: \mathbb{R} \to \mathbb{R}$ be monotone non-decreasing & define

$$F_-(x) := \sup_{y<x} F(y)$$

$$F_+(x) := \inf_{y>x} F(y)$$

Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra. Then $\exists!$ measure $\mu_F : \mathcal{B}_{\mathbb{R}} \to [0,+\infty]$ s.t.

$$\mu_F([a,b]) = F_+(b) - F_-(a), \quad \mu_F([a,b)) = F_-(b) - F_-(a), \quad \mu_F((a,b]) = F_+(b) - F_+(a), \quad \mu_F((a,b)) = F_-(b) - F_+(a)$$

$\forall\, -\infty < a < b < \infty$ & $\mu_F(\{a\}) = F_+(a) - F_-(a)$.

Pf: Define $F_-(+\infty) = \sup_{y\in\mathbb{R}} F(y)$ & $F_+(-\infty) = \inf_{y \in \mathbb{R}} F(y)$ & then define $|I|_F$ for any interval I in the obvious way. Then it's easy but annoying to check that $|I \cup J|_F = |I|_F + |J|_F$ whenever I & J are disjoint intervals, which gives finite additivity.

Now, let $\mathcal{B}_0$ be the algebra consisting of all finite unions of intervals (including infinite intervals) & define μ_0 on $\mathcal{B}_0$ by

$$\mu_0(E) = |I_1|_F + \dots + |I_n|_F$$

where $E = I_1 \cup \dots \cup I_n$ & the $I_1, \dots, I_n$ are disjoint intervals. By construction this is finitely additive & I claim μ_0 is a pre-measure.

To see this, suppose $E \in \mathcal{B}_0$ s.t. $E = \bigcup_{i=1}^{\infty} E_i$ where the $E_i \in \mathcal{B}_0$ are pairwise disjoint. We need to prove that

$$\mu_0(E) = \sum_{i=1}^{\infty} \mu_0(E_i)$$

By finite subadditivity & monotonicity, $\mu_0(E) = \mu_0\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \mu_0\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu_0(E_i)$ for all n, so $\mu_0(E) \geq \sum_{i=1}^{\infty} \mu_0(E_i)$.

So the hard part is to show that $\mu_0(E) \leq \sum_{i=1}^{\infty} \mu_0(E_i)$.

Now, here's the big idea: if we could replace the countable union by a finite union, we'd be done, since μ_0 is finitely subadditive.

How can we get a finite union: well, what we've got is a (countable) covering $\bigcup_{i=1}^{\infty} E_i$ of a set E. If the set were compact & if the covering were an open covering, we'd get to replace the countable union by a finite union since any open cover of a compact set has a finite subcover.

Of course E may not be compact & the E_i may not be open, but the idea is to manipulate things so we can replace E by a compact set and we can replace each E_i by some open set U_i.

First, we just simplify E & E_i a bit: E is a finite union of intervals; by dealing with each in turn & using finite additivity, we can reduce to the case when E is a single interval.

Also, each E_i is a finite union of intervals so, by breaking each up into a finite union of disjoint intervals, we get E as the disjoint union of countably many intervals, so we may as well assume each E_i is a single interval.

Now, by definition of μ_0, $\mu_0(E) = \sup_{[a,b]\subseteq E} \mu_0([a,b])$, so it's enough to show $\mu_0([a,b]) \le \sum_{i=1}^{\infty} \mu_0(E_i)$ for any $[a,b]\subseteq E$, and $[a,b]$ is the compact set we were looking for.

Similarly, $\mu_0(E_i) = \inf_{U\supseteq E_i} \mu_0(U)$, where U ranges over all open intervals containing E_i (i.e., U is of the form (a,b) or $(a,+\infty)$ or $(-\infty,b)$), meaning that, for any $\varepsilon>0$, $\exists\, U_i \supseteq E_i$ an open interval s.t. $\mu_0(U_i) < \mu_0(E_i)+\frac{\varepsilon}{2^i}$.

Therefore, $\sum_{i=1}^{\infty} \mu_0(U_i) \le \sum_{i=1}^{\infty}(\mu_0(E_i)+\varepsilon/2^i) = \sum_{i=1}^{\infty} \mu_0(E_i)+\varepsilon$.

In particular, $\mu_0([a,b]) \le \sum \mu_0(E_i)$ will follow if we can show $\mu_0([a,b]) \le \sum_{i=1}^{\infty} \mu_0(U_i)$ where U_i is an open interval containing E_i, and we're now reduced to the case of an open cover of a compact set.

The open cover $\bigcup U_i \supseteq [a,b]$ must have a finite subcover, so, after re-indexing, $\exists\, n$ s.t. $[a,b]\subseteq \bigcup_{i=1}^{n} U_i$.

Now we're done: by finite subadditivity (which follows from finite additivity), $\mu_0([a,b]) \le \mu_0\left(\bigcup_{i=1}^{n} U_i\right) \le \sum_{i=1}^{n} \mu_0(U_i) \le \sum_{i=1}^{\infty} \mu_0(U_i)$.

Hence, μ_0 is a premeasure & so, by Hahn-Kolmogorov, extends to a measure μ on some σ-algebra $\mathcal{B}$ containing $\mathcal{B}_0$.

Then $\mu_F = \mu|_{\mathcal{B}_{\mathbb{R}}}$ is the measure we want. ∎

Ex: When $F(x)=x$, the measure μ_F is Lebesgue measure, usually denoted by m or λ.

Ex: When $F(x)=H(x)=\begin{cases}1 & \text{if } x\ge 0\\ 0 & \text{else}\end{cases}$, the measure μ_F is Dirac measure.

Let m be Lebesgue measure & let $\mathcal{L}$ be the collection of Lebesgue measurable sets on $\mathbb{R}$.

We saw before that, for any $E \subseteq \mathbb{R}$ countable, $m(E) = m\left(\bigcup_{i=1}^{\infty}\{x_i\}\right) = \sum_{i=1}^{\infty} m(\{x_i\}) = \sum_{i=1}^{\infty} 0 = 0$

But it is possible for uncountable sets to have Lebesgue measure 0:

Prop: Let $C \subseteq [0,1]$ be the Cantor set. Then $m(C) = 0$.

Proof: Let $I_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$, $I_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$, etc. Then $C = [0,1] \setminus \bigcup_{n=1}^{\infty} I_n$, so

$$m(C) = m([0,1]) - m\left(\bigcup_{n=1}^{\infty} I_n\right) = 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1 - \frac{1/3}{1 - 2/3} = 1 - 1 = 0 .$$

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