Math 617: Days 43-44

Fourier Series & the Fundamental Solution of the Laplacian

Suppose we want to solve the Poisson equation $\Delta u = f$ on the circle.

First off, what constraints on f ensure the existence of a solution?

Well, if $f = \Delta u = \frac{d^2}{dx^2} u$, then $\int_{S^1} f\,dx = \int_{S^1} \frac{d^2}{dx^2} u\,dx = \left[\frac{du}{dx}\right]_0^1 = 0$, so f must have avg. value 0.

In fact, this is the only constraint.

To solve, write $u = \sum c_n e^{2\pi i n x}$ & $f = \sum a_n e^{2\pi i n x}$.

Then $\frac{du}{dx} = \frac{d}{dx}\left(\sum c_n e^{2\pi i n x}\right) = \sum 2\pi i n c_n e^{2\pi i n x}$

& so $\Delta u = \frac{d^2}{dx^2} u = \frac{d}{dx}\left(\sum 2\pi i n c_n e^{2\pi i n x}\right) = \sum -4\pi^2 n^2 c_n e^{2\pi i n x}$

So the rule is that $a_n = -4\pi^2 n^2 c_n$. Note that this means $a_0 = 0$, so $\int f\,dx = \hat{f}(0) = a_0 = 0$, as we knew had to be the case.

Hence, the solution to the Poisson equation is $u(x) = \sum_{n \neq 0} \frac{-a_n}{4\pi^2 n^2} e^{2\pi i n x}$.

In other words, we multiplied the n^{th} Fourier coefficient $a_n = \hat{f}(n)$ of n by $\frac{-1}{4\pi^2 n^2}$.

Define $\varphi(x) := \sum \frac{-1}{4\pi^2 n^2} e^{2\pi i n x}$, the fundamental solution of the Laplacian on S^1 (it turns out that $\varphi(x) = \frac{x(1-x)}{2} - \frac{1}{12}$).

Then we want to multiply the Fourier coefficients of φ & f to get the Fourier coefficients of u.

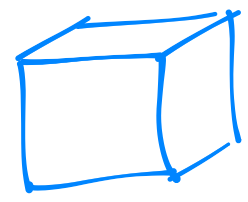
But now, as you (hopefully) saw last week, $\hat{\varphi}\hat{f} = \widehat{\varphi * f}$; i.e., the product of the Fourier transforms of φ & f is the Fourier transform of the convolution of φ & f.

In other words, $\varphi * f = u$ & the fundamental sol'n of the Laplacian is exactly the fn. φ s.t. $\Delta(\varphi * f) = f$ for any $f: S^1 \to \mathbb{R}$ w/ avg. value 0.

More generally, for any f, $\Delta(\varphi * f) = f - \hat{f}(0)$... the mapping $f \mapsto \varphi * f$ is also sometimes called Green's operator.

Since constant functions are the only harmonic functions on S^1, we can also think of $\hat{f}(0)$ as the harmonic part of f.

Vector differential equations on the 3-torus

Now consider the 3-dimensional torus $T^3 = S^1\times S^1\times S^1$, which we can just think of as the unit cube 

w/ periodic boundary conditions.

Given a vector field V on T^3, we can try to solve eqns like

$$\nabla u = V \quad \text{or} \quad \nabla\times U = V \quad \text{or} \quad \nabla\cdot U = f.$$

Now, we can extend the theory of Fourier series to both functions & vector fields on T^3. It turns out that, for

$f\in L^2(T^3)$,

$$f(\vec{x}) = \sum_{\vec{n}\in\mathbb{Z}^3} a_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}}$$

where $\vec{x}=(x_1,x_2,x_3)\in S^1\times S^1\times S^1$ is a point in T^3 &, for $\vec{n}=(n_1,n_2,n_3)$, $\vec{n}\cdot\vec{x}$ just means the std. dot product

$$\vec{n}\cdot\vec{x} = n_1x_1+n_2x_2+n_3x_3.$$

Convolution is defined in the usual way: $(f*g)(x) = \int_{T^3} f(\vec{y})\,g(\vec{x}-\vec{y})\,d\vec{y}$ & it's easy to check that

$e^{2\pi i\vec{m}\cdot\vec{x}} * e^{2\pi i\vec{n}\cdot\vec{x}} = 0$ if $\vec{m}\neq\vec{n}$ & that $e^{2\pi i\vec{n}\cdot\vec{x}} * e^{2\pi i\vec{n}\cdot\vec{x}} = e^{2\pi i\vec{n}\cdot\vec{x}}$.

Hence, if $f=\sum a_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}}$ & $g=\sum b_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}}$, then $f*g = \sum a_{\vec{n}} b_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}}$.

Vector fields also have Fourier series: say that $V(\vec{x}) = (v_1(\vec{x}), v_2(\vec{x}), v_3(\vec{x}))$. Each v_j is just a function, so

it has a Fourier series: $v_j(\vec{x}) = \sum_{\vec{n}\in\mathbb{Z}^3} c^j_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}}$ & $V(\vec{x}) = \left(\sum c^1_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}},\ \sum c^2_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}},\ \sum c^3_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}}\right)$

$= \sum_{\vec{n}\in\mathbb{Z}^3}\sum_{j=1}^{3} c^j_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}}\,\vec{e}_j$, where $\vec{e}_j$ is the j^{th} std basis vector.

Writing $\vec{c}_{\vec{n}} = (c^1_{\vec{n}}, c^2_{\vec{n}}, c^3_{\vec{n}})$ & $\vec{e} = (\vec{e}_1,\vec{e}_2,\vec{e}_3)$, I'll adopt the shorthand $V(\vec{x}) = \sum_{\vec{n}\in\mathbb{Z}^3} \left(\vec{c}_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}}\right)\cdot\vec{e}$.

Let's look at what the various differential operators do to $f = \sum a_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}}$ & $V = \sum\left(\vec{c}_{\vec{n}}\, e^{2\pi i\vec{n}\cdot\vec{x}}\right)\cdot\vec{e}$:

$$\nabla f = \frac{\partial f}{\partial x_1}\vec{e}_1 + \frac{\partial f}{\partial x_2}\vec{e}_2 + \frac{\partial f}{\partial x_3}\vec{e}_3.$$

Now, $\frac{\partial}{\partial x_j}\left(e^{2\pi i\vec{n}\cdot\vec{x}}\right) = \frac{\partial}{\partial x_j}\left(e^{2\pi i(n_1x_1+n_2x_2+n_3x_3)}\right) = \frac{\partial}{\partial x_j}\left(e^{2\pi i n_1x_1} e^{2\pi i n_2x_2} e^{2\pi i n_3x_3}\right) = 2\pi i n_j\, e^{2\pi i n_1x_1} e^{2\pi i n_2x_2} e^{2\pi i n_3x_3}$

$= 2\pi i n_j\, e^{2\pi i\vec{n}\cdot\vec{x}}$, So $\nabla f = \sum_{\vec n\in\mathbb{Z}^3}\left(2\pi i n_1 a_{\vec n} e^{2\pi i \vec n\cdot\vec x}\,\vec e_1 + 2\pi i n_2 a_{\vec n} e^{2\pi i\vec n\cdot\vec x}\,\vec e_2 + 2\pi i n_3 a_{\vec n} e^{2\pi i\vec n\cdot\vec x}\,\vec e_3\right)$

$$= \sum_{\vec n\in\mathbb{Z}^3} 2\pi i\,\vec n\, a_{\vec n}\, e^{2\pi i\vec n\cdot\vec x}\cdot\vec e$$

In turn, $\nabla\cdot V = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \sum_{j=1}^{3}\sum_{\vec n\in\mathbb{Z}^3} 2\pi i n_j c^j_{\vec n} e^{2\pi i\vec n\cdot\vec x} = \sum_{\vec n\in\mathbb{Z}^3} 2\pi i\,\vec n\cdot\vec c_{\vec n}\, e^{2\pi i\vec n\cdot\vec x}$

Hence, we see that $\Delta f = \nabla\cdot(\nabla f) = \nabla\cdot\left(\sum_{\vec n\in\mathbb{Z}^3} 2\pi i\vec n\, a_{\vec n} e^{2\pi i\vec n\cdot\vec x}\cdot\vec e\right) = \sum_{\vec n\in\mathbb{Z}^3} -4\pi^2\,\vec n\cdot(\vec n\, a_{\vec n})\, e^{2\pi i\vec n\cdot\vec x} = \sum_{\vec n\in\mathbb{Z}^3} -4\pi^2 a_{\vec n}|\vec n|^2 e^{2\pi i\vec n\cdot\vec x}$.

Therefore, if we let $\varphi(\vec x) = \frac{-1}{4\pi^2}\sum_{\vec n\neq\vec 0}\frac{1}{|\vec n|^2} e^{2\pi i\vec n\cdot\vec x}$, then

$$\Delta(\varphi * f) = f - a_{\vec 0}$$

& φ is the Fundamental Solution of the (Scalar) Laplacian on T^3.

In particular, for any f w/ $\int_{T^3} f = 0$, the solution to the Poisson eqn. $\Delta u = f$ is given by $u = \varphi * f = \frac{-1}{4\pi^2}\sum_{\vec n\neq\vec 0}\frac{a_{\vec n}}{|\vec n|^2} e^{2\pi i\vec n\cdot\vec x}$.

Finally, the effect of curl is:

$$\nabla\times V = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\vec e_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\vec e_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\vec e_3 = \sum_{\vec n\in\mathbb{Z}^3}\Big[2\pi i(n_2 c^3_{\vec n} - n_3 c^2_{\vec n})\vec e_1 + 2\pi i(n_3 c^1_{\vec n} - n_1 c^3_{\vec n})\vec e_2 + 2\pi i(n_1 c^2_{\vec n} - n_2 c^1_{\vec n})\vec e_3\Big] e^{2\pi i\vec n\cdot\vec x}$$

$$= \sum_{\vec n\in\mathbb{Z}^3} 2\pi i\,\vec n\times\vec c_{\vec n}\, e^{2\pi i\vec n\cdot\vec x}\cdot\vec e$$

Now, notice that $\ker(\nabla) = \{f : a_{\vec n} = 0 \text{ for } \vec n\neq\vec 0,\ a_{\vec 0} \text{ arbitrary}\} = \{\text{constant functions}\}$

$\operatorname{im}(\nabla) = \{V : \vec c_{\vec n} = \lambda_{\vec n}\vec n \text{ for } \vec n\neq\vec 0,\ \vec c_{\vec 0} = \vec 0\}$

$\ker(\nabla\cdot) = \{V : \vec n\cdot\vec c_{\vec n} = 0 \text{ for } \vec n\neq\vec 0,\ \vec c_{\vec 0} \text{ arbitrary}\}$

$\operatorname{im}(\nabla\cdot) = \{f : a_{\vec 0} = 0\}$

$\ker(\nabla\times) = \{V : \vec n\times\vec c_{\vec n} = \vec 0 \text{ for } \vec n\neq\vec 0,\ \vec c_{\vec 0} \text{ arbitrary}\} = \{V : \vec c_{\vec n} = \lambda_{\vec n}\vec n \text{ for } \vec n\neq\vec 0,\ \vec c_{\vec 0} \text{ arbitrary}\}$

$\operatorname{im}(\nabla\times) = \{V : \vec c_{\vec n}\cdot\vec n = \vec 0 \text{ for } \vec n\neq\vec 0,\ \vec c_{\vec 0} = \vec 0\}$

So now we know exactly what the conditions are that allow an eqn like $\nabla u = V$, $\nabla\times U = V$, $\nabla\cdot U = f$, $\Delta U = V$ to be solved.

For example, $\nabla u = V$ is solvable if $V\in\operatorname{im}(\nabla)$, meaning if $\vec c_{\vec n} = \lambda_{\vec n}\vec n$ & $\vec c_{\vec 0} = \vec 0$, & the soln is

$$u = \sum_{\vec n\in\mathbb{Z}^3}\frac{\lambda_{\vec n}}{2\pi i} e^{2\pi i\vec n\cdot\vec x}$$

We can more suggestively write this as $u = \nabla\cdot(\varphi * V)$ since

$$\varphi * V = \frac{-1}{4\pi^2}\sum_{\vec{n}\neq\vec{0}} \frac{\vec{c}_{\vec{n}}}{|\vec{n}|^2} e^{2\pi i\vec{n}\cdot\vec{x}}$$

&

$$\nabla\cdot(\varphi * V) = \frac{-1}{4\pi^2}\sum_{\vec{n}\neq\vec{0}} \frac{2\pi i}{|\vec{n}|^2}\vec{n}\cdot\vec{c}_{\vec{n}} e^{2\pi i\vec{n}\cdot\vec{x}} = \frac{-1}{2\pi i}\sum \frac{1}{|\vec{n}|^2}\vec{n}\cdot(\lambda_{\vec{n}}\vec{n}) e^{2\pi i\vec{n}\cdot\vec{x}} = \frac{1}{2\pi i}\sum \lambda_{\vec{n}} e^{2\pi i\vec{n}\cdot\vec{x}}.$$

So $\nabla(\nabla\cdot(\varphi * V)) = V$.

Now, the condition $\vec{c}_{\vec{n}} = \lambda_{\vec{n}}\vec{n}$ is a little weird, but notice that this is the same as the condition that $V \in \ker(\nabla\times)$, so really the condition is: $\nabla u = V$ is solvable $\Leftrightarrow$ V is curl-free & $\int_{T^3} V = \vec{0}$ & the sol'n is $u = \nabla\cdot(\varphi * V)$.

Similarly, $\nabla\cdot u = f$ has a sol'n if $f \in \mathrm{im}(\nabla\cdot)$, meaning $a_{\vec{0}} = \vec{0}$ (or, equivalently, $\int_{T^3} f = 0$), & the sol'n is given by

$$u = \sum_{\vec{n}\neq\vec{0}} \vec{c}_{\vec{n}} e^{2\pi i\vec{n}\cdot\vec{x}}$$

for any choice of $\vec{c}_{\vec{n}}$ s.t. $\vec{n}\cdot\vec{c}_{\vec{n}} = \frac{a_{\vec{n}}}{2\pi i}$ .... for example, $\vec{c}_{\vec{n}} = \frac{1}{2\pi i}\frac{a_{\vec{n}}\vec{n}}{|\vec{n}|^2}$, which is the Fourier coefficient of $\nabla(\varphi * f)$. So $\nabla\cdot(\nabla(\varphi * f)) = f$

For $\nabla\times u = V$, we have a sol'n if $V \in \mathrm{im}(\nabla\times)$, meaning $\vec{c}_{\vec{n}}\cdot\vec{n} = 0$ for $\vec{n}\neq\vec{0}$ & $\vec{c}_{\vec{0}} = \vec{0}$. Of course, this just means V is divergence-free & has $\int_{T^3} V = \vec{0}$. The solution is

$$u = \sum_{\vec{n}\neq\vec{0}} \vec{d}_{\vec{n}} e^{2\pi i\vec{n}\cdot\vec{x}}$$

for any $\vec{d}_{\vec{n}}$ s.t. $\vec{n}\times\vec{d}_{\vec{n}} = \frac{\vec{c}_{\vec{n}}}{2\pi i}$ . . . for example, $\vec{d}_{\vec{n}} = -\frac{1}{2\pi i}\frac{\vec{n}\times\vec{c}_{\vec{n}}}{|\vec{n}|^2}$, which is the Fourier coefficient of

$-\nabla\times(\varphi * V)$. So $\nabla\times(-\nabla\times(\varphi * V)) = V$

Finally, for $\Delta u = V$, it's easy to check that $\Delta\left(\sum \vec{d}_{\vec{n}} e^{2\pi i\vec{n}\cdot\vec{x}}\right) = -4\pi^2\sum_{\vec{n}\in\mathbb{Z}^3} |\vec{n}|^2\vec{d}_{\vec{n}} e^{2\pi i\vec{n}\cdot\vec{x}}$

so clearly $u = \varphi * V$ is a sol'n so long as V has $\vec{c}_{\vec{0}} = \vec{0}$.

One more thing: the collection of vector fields on T^3 has an orthogonal decompositions Hodge decomposition

$$VF(T^3) = \ker(\nabla\times)\oplus\mathrm{im}(\nabla\times) = \mathrm{im}(\nabla)\oplus\ker(\nabla\cdot) = \mathrm{im}(\nabla\times)\oplus\mathrm{Har}(T^3)\oplus\mathrm{im}(\nabla)$$

where $\mathrm{Har}(T^3) = \ker(\nabla\cdot)\cap\ker(\nabla\times)$ is the harmonic fields. The Laplacian $\Delta: VF(T^3)\to VF(T^3)$ sends $\mathrm{im}(\nabla\times)$ bijectively to itself, kills $\mathrm{Har}(T^3)$, & takes $\mathrm{im}(\nabla)$ bijectively to itself.

This is all easy to see. The L^2 inner product on $VF(T^3)$ is just $\langle u,v\rangle = \int_{T^3} u \cdot v$.

But then if, say, $u \in \ker(Dx)$ & $v \in \mathrm{im}(Dx)$, then $u = \vec{c}_{\vec{0}} + \sum_{\vec{n} \neq \vec{0}} \lambda_{\vec{n}} \vec{n} e^{2\pi i \vec{n} \cdot \vec{x}} \cdot \vec{e}$ & $v = \sum_{\vec{n} \neq \vec{0}} \vec{c}_{\vec{n}} e^{2\pi i \vec{n} \cdot \vec{x}} \cdot \vec{e}$

w/ $\vec{c}_{\vec{n}} \cdot \vec{n} = 0 \ \forall \ \vec{n} \neq \vec{0}$, so

$$\langle u, v\rangle = \int_{T^3} u \cdot v = \int_{T^3} \sum \lambda_{\vec{n}} \vec{n} \cdot \vec{c}_{\vec{n}} = \int_{T^3} 0 = 0 .$$