# Math 617: Days 40-41

## Fourier Analysis

Consider the circle $S^1 = \mathbb{R}/\mathbb{Z}$, which we can think of as the unit interval $[0,1]$ w/ periodic boundary conditions.

Thought of in this way, the circle has an obvious Lebesgue measure $m$, & we can look at $L^2(S^1, \mathcal{L}, m)$, the square-integrable measurable functions $S^1 \to \mathbb{C}$, which is a Hilbert space.

The claim is that this is a separable Hilbert space & that the collection $(e_n)_{n\in\mathbb{Z}}$ where $e_n(x) := e^{2\pi i n x}$ is an ON basis.

**Lemma:** $(e_n)_{n\in\mathbb{Z}}$ is an ON system.

**Pf:** Let $j,k \in \mathbb{Z}$. Then

$$\langle e_j, e_k\rangle = \int_{S^1} e^{2\pi i j x}\overline{e^{2\pi i k x}}\,dm(x) = \int_0^1 e^{2\pi i (j-k)x}\,dx = \frac{e^{2\pi i (j-k)x}}{2\pi i (j-k)}\Big]_0^1 = \frac{e^{2\pi i (j-k)}}{2\pi i (j-k)} - \frac{1}{2\pi i (j-k)}$$

$$= \begin{cases} \dfrac{0}{2\pi i (j-k)} = 0 & \text{if } j \neq k \\ \lim\limits_{t\to 0} \dfrac{e^{2\pi i t}}{2\pi i t} = 1 & \text{if } j = k. \end{cases}$$

$\square$

Now, by ② from the Proposition, to prove $(e_n)_{n\in\mathbb{Z}}$ is an ON basis, it suffices to show that the algebraic span of the $e_n$ are dense in $L^2(S^1, \mathcal{L}, m)$. In other words, the finite sums $\sum_{n=j}^{k} c_n e^{2\pi i n x}$ (called trigonometric polynomials since $e^{2\pi i n x} = \left(e^{2\pi i x}\right)^n = \left(\cos(2\pi x) + i\sin(2\pi x)\right)^n = \sum_{k=1}^{n} i^n \binom{n}{k}\cos^k(2\pi x)\sin^{n-k}(2\pi x)$) are dense in $L^2(S^1, \mathcal{L}, m)$.

One approach is to observe that the trigonometric polynomials separate points (i.e., for any $x, y \in S^1$, $\exists\, n \in \mathbb{Z}$ s.t. $e_n(x) \neq e_n(y)$) & are closed under addition, multiplication, & complex conjugation. Then the Stone-Weierstrass Thm implies the trigonometric polynomials are dense in the continuous functions & hence dense in $L^2$.

Another approach is more hands-on...

We've seen that the simple functions w/ finite measure support are dense in any $L^p$; since $S^1$ has finite measure $m(S^1) = 1$, the finite measure support condition is vacuous, so it suffices to show we can approximate simple functions arbitrarily well by trigonometric polynomials.

In fact, since measurable sets are approximated arbitrarily well by finite unions of intervals, it suffices to show that $\chi_I$ where $I \subseteq [0,1]$ is an interval is approximated arbitrarily well by trig polynomials.

So now, let $I$ be an interval w/ endpoints $a$ & $b$.

Let $s_n(x) = \sum_{k=-n}^{n} \hat{\chi}_I(k)\, e^{2\pi i k x} = \sum_{k=-n}^{n} \left( \int_{S^1} \chi_I(t)\, e^{-2\pi i k t}\, dt \right) e^{2\pi i k x} = \int_{S^1} \chi_I(t) \sum_{k=-n}^{n} e^{2\pi i k (x-t)}\, dt$

$$= \int_{S^1} \chi_I(x-t) \sum_{k=-n}^{n} e^{2\pi i k t}\, dt = \int_{x-b}^{x-a} \sum_{k=-n}^{n} e^{2\pi i k t}\, dt = \int_{x-b}^{x-a} D_n(t)\, dt$$

where $D_n(t) := \sum_{k=-n}^{n} e^{2\pi i k t}$ is the Dirichlet kernel.

Now, the goal is to show $s_n \to \chi_I$ in $L^2$... to do so, we will actually show that the Cesàro means

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k$$

converge to $\chi_I$. So consider

$$\sigma_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} s_k(x) = \frac{1}{n+1} \sum_{k=0}^{n} \int_{x-b}^{x-a} D_k(t)\, dt = \int_{x-b}^{x-a} \frac{1}{n+1} \sum_{k=0}^{n} D_k(t)\, dt = \int_{x-b}^{x-a} F_n(t)\, dt$$

where $F_n(t) := \frac{1}{n+1} \sum_{k=0}^{n} D_k(t)$ is the Fejér kernel.

It will suffice to show that $|\sigma_n(x) - \chi_I(x)| < \varepsilon$ for all sufficiently large $n$, since this means

$$\|\sigma_n - \chi_I\|_{L^2} = \sqrt{\int_{S^1} |\sigma_n - \chi_I|^2\, dm} \le \sqrt{\int_{S^1} \varepsilon^2\, dm} = \sqrt{\varepsilon^2} = \varepsilon.$$

Lemma: $F_n(x) = \frac{1}{n+1} \frac{\sin^2\left(\frac{2\pi(n+1)x}{2}\right)}{\sin^2\left(\frac{2\pi x}{2}\right)} = \frac{1}{n+1} \frac{\sin^2(\pi(n+1)x)}{\sin^2(\pi x)}$ $\forall\, n \in \mathbb{N}$.

Proof: Calculation. $\boxtimes$

It will turn out in what follows to identify $S^1$ with $[-1/2, 1/2)$ instead of $[0,1)$... of course, this is just a translation.

Prop: ① $\int_{S^1} F_n(x)\, dm(x) = \int_{-1/2}^{1/2} F_n(x)\, dx = 1$

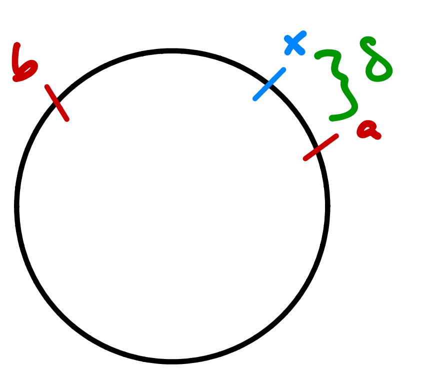
② $F_n(x) \ge 0$ $\forall\, x \in [-1/2, 1/2]$

③ For each $\delta > 0$, $\lim_{n\to\infty} \int_{|x|>\delta} F_n(x)\, dx = 0$.

But the result now follows: Let $x \in S^1$ & let $\varepsilon > 0$. Let $\delta = \min\{b-x, x-a\}$ & choose $N$ s.t.

$$n \ge N \Rightarrow \int_{|t|>\delta} F_n(t)\, dt < \varepsilon.$$

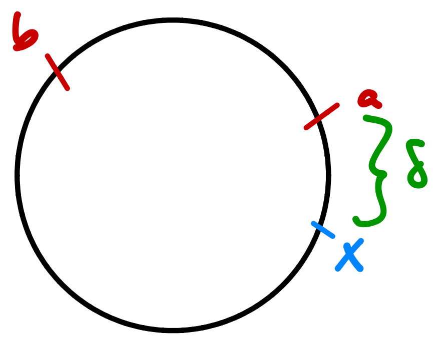
① If $x \in I$, then $[x-b, x-a] \supseteq \{t : |t| \le \delta\}$, so, for $n \ge N$,




$$\sigma_n(x) = \int_{x-b}^{x-a} F_n(t)\, dt \ge \int_{|t|\le\delta} F_n(t)\, dt = 1 - \int_{|t|>\delta} F_n(t)\, dt > 1 - \varepsilon$$

and hence $|\sigma_n(x) - \chi_I(x)| = \chi_I(x) - \sigma_n(x) = 1 - \sigma_n(x) < 1 - (1-\varepsilon) = \varepsilon$.

② OTOH, if $x \notin I$, then $[x-b, x-a] \subseteq \{t : |t| > \delta\}$, so, for $n \geq N$

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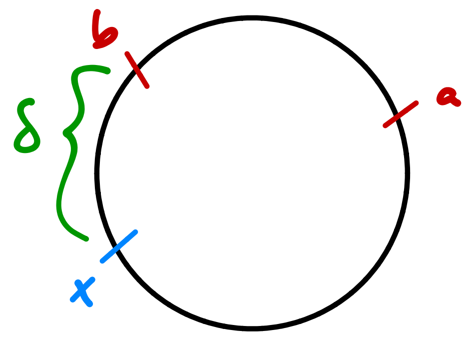


$$\sigma_n(x) = \int_{x-b}^{x-a} F_n(t)\,dt \leq \int_{|t|>\delta} F_n(t)\,dt < \varepsilon,$$

so $|\sigma_n(x) - \chi_I(x)| = \sigma_n(x) - 0 = \sigma_n(x) < \varepsilon$.

So either way, we've succeeded in showing the $\sigma_n$ & hence then $s_n$ converge to $\chi_I$, meaning that the $\{e_n\}_{n\in\mathbb{Z}}$

really are a basis, called the Fourier basis.

Hence, we define the Fourier transform $\hat{f}: \mathbb{Z} \to \mathbb{C}$ of a function $f \in L^2(S^1, \mathcal{L}, m)$ by

$$\hat{f}(n) := \langle f, e_n \rangle = \int_{S^1} f(x) e^{-2\pi i n x}\,dx.$$

From our earlier work we get the Parseval identity

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \|f\|_{L^2}^2 = \int_{S^1} |f(x)|^2\,dx$$

and the inversion formula

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n) e_n.$$

Sometimes called Plancherel's Thm

Proof of Prop: ① $\displaystyle\int_{-1/2}^{1/2} F_n(x)\,dx = \int_{-1/2}^{1/2} \frac{1}{n+1}\sum_{k=0}^{n}\sum_{m=-k}^{k} e^{2\pi i m x}\,dx = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{m=-k}^{k} \underbrace{\int_{-1/2}^{1/2} e^{2\pi i m x}\,dx}_{0 \text{ if } m\neq 0,\ 1 \text{ else}} = \frac{1}{n+1}\sum_{k=0}^{n} 1 = 1$

② $\displaystyle F_n(x) = \frac{1}{n+1}\,\frac{\sin^2(\pi(n+1)x)}{\sin^2(\pi x)} \geq 0$

③ For $|x| > \delta$, $\displaystyle\frac{1}{\sin^2(\pi x)} < \frac{1}{\sin^2(\pi\delta)}$, so $\displaystyle 0 \leq F_n(x) \leq \frac{1}{n+1}\,\frac{1}{\sin^2(\pi\delta)}$, which converges uniformly to 0, so the integral converges as well. ∎