Math 617: Days 20-21

We're now ready for the full change of variables theorem:

Thm (Change of Variables): Let $\Omega \subseteq \mathbb{R}^n$ be open & let $G: \Omega \to \mathbb{R}^n$ be injective & continuously differentiable, w/ continuously differentiable inverse on its image (i.e., a C^1 diffeomorphism). If f is measurable on $G(\Omega)$, then $f \circ G$ is Lebesgue measurable on Ω. Moreover, if $f \geq 0$ or $f \in L^1(G(\Omega), m)$, then

$$\int_{G(\Omega)} f(y)\, dm(y) = \int_\Omega f \circ G(x)\, |\det DG|\, dm(x).$$

Proof: Since G & G^{-1} both continuous, they send Borel sets to Borel sets, so G is a measurable morphism & $f \circ G$ is measurable.

Now, we know from HW 3#2 that, for any measurable morphism $\varphi: X \to Y$ w/ $\varphi_* \mu(E) = \mu(\varphi^{-1}(E))$ $\forall$ $E \subseteq Y$ measurable,

$$\int_X (f \circ \varphi)\, d\mu = \int_Y f\, d\varphi_* \mu.$$

Now, let $\varphi = G^{-1}$, let $X = G(\Omega)$, let $Y = \Omega$, and let $\mu = m$. Then this says

$$\int_{G(\Omega)} f(y)\, dm(y) = \int_{G(\Omega)} f \circ G \circ G^{-1}(y)\, dm(y) = \int_\Omega (f \circ G)\, d(G^{-1}_* m)(x),$$

so this boils down to showing that $\int_\Omega (f \circ G)\, d(G^{-1}_* m)(x) = \int_\Omega (f \circ G)\, |\det D_x G|\, dm(x)$.

Now, $G^{-1}_* m(E) = m((G^{-1})^{-1}(E)) = m(G(E))$ $\forall$ $E \subseteq \Omega$ measurable, so the theorem is true given the following 2 lemmas. ∎

Lemma 1: Let $G: \Omega \to \mathbb{R}^n$ be a C^1 diffeomorphism onto its image. Then for all measurable $E \subseteq \Omega$,

$$(G^{-1}_* m)(E) = m(G(E)) = \int_E |\det D\varphi|\, dm(x).$$

Lemma 2: If $h: X \to [0, +\infty]$ is measurable on $(X, \mathcal{M}, \mu)$ & we define

$$\nu(E) := \int_E h\, d\mu \quad \forall\ E \in \mathcal{M},$$

then ν is a measure on $(X, \mathcal{M})$ &, for any measurable f, $\int_X f\, d\nu = \int_X f h\, d\mu$.

(h is called a *density* for ν w.r.t. μ or, as we'll see soon, the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$)

Proof of Lemma 2: We saw on Day 11 (where we called it μ_h) that ν is a measure.

Now, assume first that $f = \sum_{i=1}^n c_i \chi_{E_i}$ is simple. Then

$$\int_X f\,d\nu = \int_X \sum_{i=1}^n c_i \chi_{E_i}\,d\nu = \sum_{i=1}^n c_i \int_X \chi_{E_i}\,d\nu = \sum_{i=1}^n c_i\,\nu(E_i) = \sum_{i=1}^n c_i \int_{E_i} h\,d\mu = \sum_{i=1}^n c_i \int_X \chi_{E_i} h\,d\mu$$

$$= \int_X \sum_{i=1}^n c_i \chi_{E_i}\, h\,d\mu$$

$$= \int_X f h\,d\mu,$$

so the result is true for simple functions. In turn, by applying the Monotone Convergence Theorem to the limiting sequence of increasing simple functions, it's true for $f \geq 0$. But then, by applying to positive & negative parts, it's true for real valued f, & by applying to real & imaginary parts, it's true for complex valued f. ∎

Proof of Lemma 1 (sketch): Let $x \in \Omega$ & let $\varepsilon > 0$. Then, since G is C^1, it can be approximated by its linearization DG on sufficiently small nbhds of x. In particular, for Q_x a small enough cube centered on x, $m(G(Q)) \leq (1+\varepsilon)\, m(D_x G(Q)) = (1+\varepsilon)\,|\det D_x G|\, m(Q)$ using linear change of variables.

Now, if Q is a cube of any size, we can break it up into a union of little cubes of the form above which only intersect on their boundaries.

So then $m(G(Q)) = \sum_{i=1}^n m(G(Q_i)) \leq (1+\varepsilon) \sum_{i=1}^n |\det D_{x_i} G|\, m(Q_i)$.

Taking limits as $\varepsilon \to 0$ gives $m(G(Q)) \leq \int_Q |\det D_x G|\, dm(x)$.

In turn, any open $U \subseteq \Omega$ can be written as a countable union of cubes intersecting only at their boundaries, so

$$m(G(U)) = m\left(\bigcup_{i=1}^\infty G(Q_i)\right) \leq \sum_{i=1}^\infty m(G(Q_i)) \leq \sum_{i=1}^\infty \int_{Q_i} |\det D_x G|\, dm(x) = \int_U |\det D_x G|\, dm(x).$$

Some fancy footwork (see Thm 2.47 in book) shows that the same estimate

$$m(G(E)) \leq \int_E |\det D_x G|\, dm(x)$$

holds for any Borel set $E \subseteq \Omega$.

Of course, for little cubes $m(G(Q)) \geq (1-\varepsilon)\,|\det D_x G|\, m(Q)$, so we have the reverse inequality as well, and

$$m(G(E)) = \int_E |\det D_x G|\, dm(x),$$

as desired. ∎

See §2.7 in Folland for a nice application to integrating the Gaussian density & computing volumes of spheres.