Math 617: Day 9

Translating & scaling of Lebesgue measure do what you expect:

Prop: Let $E \in \mathcal{L}$ & let $s, r \in \mathbb{R}$. Then the sets

$$E + s = \{x + s : x \in E\} \quad \& \quad rE = \{rx : x \in E\}$$

are Lebesgue measurable & $m(E+s) = m(E)$, $m(rE) = |r|\, m(E)$.

Proof: Since this is true of lengths of intervals, it's true on all elementary sets & hence on all Borel sets (by Hahn-Kolmogorov).

But, by the following, Lebesgue measurable sets are unions of Borel sets & null sets, so we're done. ∎

Def: A **Radon measure** μ on $\mathbb{R}$ is a Borel measure (meaning it is defined on $\mathcal{B}_{\mathbb{R}}$) s.t.

1. (Local finiteness) $\mu(K) < \infty$ $\forall$ compact $K \subseteq \mathbb{R}$.
2. (Inner regularity) $\mu(E) = \sup_{K \subseteq E,\, K \text{ cpt}} \mu(K)$ $\forall$ $E \in \mathcal{B}_{\mathbb{R}}$
3. (Outer regularity) $\mu(E) = \inf_{U \supseteq E,\, U \text{ open}} \mu(U)$ $\forall$ $E \in \mathcal{B}_{\mathbb{R}}$.

Prop: If μ_F is the Lebesgue-Stieltjes measure associated to any monotone nondecreasing $F: \mathbb{R} \to \mathbb{R}$, then μ_F is a Radon measure.

Proof: Similar to the argument we gave in the proof of the existence of the Lebesgue-Stieltjes measure in the case E was a single interval.

(See Thm 1.18 in the book for details). ∎

Thm: Conversely, if μ is a Radon measure, $\exists$ monotone nondecreasing $F: \mathbb{R} \to \mathbb{R}$ s.t. $\mu = \mu_F$.

Proof: HW. ∎

Def: If X is a topological space,
- a G_δ set in X is a countable intersection $\bigcap_{n=1}^{\infty} U_n$ of open sets $U_n \subseteq X$
- an F_σ set is a countable union $\bigcup_{n=1}^{\infty} F_n$ of closed sets $F_n \subseteq X$.

Prop: Let μ_F be a Lebesgue-Stieltjes measure & let $\mathcal{M}$ be the μ_F-measurable sets. Let $E \subseteq \mathbb{R}$. Then the following are equivalent:

1. $E \in \mathcal{M}$
2. $E = U \setminus N$, where U is G_δ & $\mu_F(N) = 0$
3. $E = K \cup N'$ where K is F_σ & $\mu_F(N') = 0$.

Pf: ② ⇒ ① & ③ ⇒ ① because μ_F is complete.

OTOH, if $E \in \mathcal{M}$ & $\mu_F(E) < \infty$, then the previous Prop implies that, for any $n \in \mathbb{N}$, $\exists$ open $U_n \supseteq E$ & compact $K_n \subseteq E$ s.t.

$$\mu_F(U_n) - \tfrac{1}{2^n} \le \mu_F(E) \le \mu_F(K) + \tfrac{1}{2^n}.$$

Then, letting $U = \bigcap_{n=1}^{\infty} U_n$ & $K = \bigcup_{n=1}^{\infty} K_n$, we see that $K \subseteq E \subseteq U$ & $\mu_F(K) = \mu_F(E) = \mu_F(U)$, so $\mu_F(U \setminus E) = \mu_F(U) - \mu_F(E) = 0$

& $\mu_F(E \setminus K) = \mu_F(E) - \mu_F(K) = 0$ (here we're using $\mu_F(E) < \infty$), so we can let $N = U \setminus E$ & $N' = E \setminus K$.

The case $\mu_F(E) = \infty$ is (maybe) HW. ∎