Math 617: Day 6

Carathéodory's Thm: If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra & the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Pf: By construction, $\emptyset \in \mathcal{M}$ & $\mathcal{M}$ is closed under complementation. We leave the fact that $\mathcal{M}$ is closed under countable unions as a reading exercise, so $\mathcal{M}$ is a σ-algebra. To see that $\mu^*|_{\mathcal{M}}$ is a measure, we already know $\mu^*(\emptyset)=0$, so we only need to prove countable additivity.

Suppose $E_1, E_2, \ldots \in \mathcal{M}$ are disjoint. We want to show that

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu^*(E_i)$$

By subadditivity, it suffices to show $\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum \mu^*(E_i)$. In turn, since $\mu^*\left(\bigcup_{i=1}^{n} E_i\right) \leq \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right)$, this will follow from showing $\mu^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu^*(E_i)$ $\forall\, n \in \mathbb{N}$. We prove this by induction:

Base Case: Obviously, $\mu^*(E_1) = \mu^*(E_1)$

Inductive Step: Assume $\mu^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu^*(E_i)$. Then, since $\bigcup_{i=1}^{n} E_i$ is μ^*-measurable,

$$\mu^*\left(\bigcup_{i=1}^{n+1} E_i\right) = \mu^*\left(\left(\bigcup_{i=1}^{n+1} E_i\right) \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + \mu^*\left(\left(\bigcup_{i=1}^{n+1} E_i\right) \cap \left(\bigcup_{i=1}^{n} E_i\right)^c\right)$$

$$= \mu^*\left(\bigcup_{i=1}^{n} E_i\right) + \mu^*(E_{n+1}) = \sum_{i=1}^{n} \mu^*(E_i) + \mu^*(E_{n+1}) = \sum_{i=1}^{n+1} \mu^*(E_i), \text{ as desired.}$$

$\blacksquare$

Def: A collection $\mathcal{A}$ of subsets of X is called an *algebra* if $\mathcal{A}$ is closed under taking complements & *finite* unions.

If $\mathcal{A}$ is an algebra on X, a map $\mu_0 : \mathcal{A} \to [0,\infty]$ is called a *premeasure* if

1. $\mu_0(\emptyset) = 0$
2. If $E_1, E_2, \ldots \in \mathcal{A}$ are disjoint sets s.t. $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$, then $\mu_0\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=0}^{\infty} \mu_0(E_i)$

Note: Since $\mathcal{A}$ is only closed under finite unions, it may well not be the case that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.

Ex: The elementary sets on $\mathbb{R}$ form an algebra (but not a σ-algebra), & elementary measure gives a premeasure.

Now, if μ_0 is a premeasure on an algebra $\mathcal{A}$, then we can define an outer measure

$$\mu^*(E) := \inf\left\{\sum_{i=1}^{\infty} \mu_0(E_i) : E_i \in \mathcal{A},\ E \subseteq \bigcup_{i=1}^{\infty} E_i\right\}$$

Hahn-Kolmogorov Thm: Every premeasure μ_0 on an algebra $\mathcal{A}$ on X can be extended to a measure μ on X.

Pf: Let μ^* be the outer measure defined above, let $\mathcal{M}$ be the collection of μ^*-measurable sets, and let $\mu = \mu^*|_{\mathcal{M}}$. Then Carathéodory's Thm implies that $\mathcal{M}$ is a σ-algebra & μ is a measure.

Of course, to see that μ actually *extends* μ_0, we need to show that (i) $\mathcal{A} \subseteq \mathcal{M}$ & (ii) that $\mu|_{\mathcal{A}} = \mu_0$.

To see (i), let $E \in \mathcal{A}$ & let $C \subseteq X$ be arbitrary. We need to show that

$$\mu^*(C) = \mu^*(C \cap E) + \mu^*(C \cap E^c)$$

& of course it suffices to show $\mu^*(C) \geq \mu^*(C \cap E) + \mu^*(C \cap E^c)$ since subadditivity implies the other inequality.

We might as well assume $\mu^*(C) < \infty$, since the inequality is obviously true otherwise.

Now, recall that $\mu^*(C) = \inf\{ \sum_1^\infty \mu_0(E_i) : E_i \in \mathcal{A}, \bigcup_1^\infty E_i \supseteq C\}$ so, for any $\varepsilon > 0$ $\exists$ $E_1, E_2, \ldots \in \mathcal{A}$ s.t.

$$\bigcup_1^\infty E_i \supseteq C \quad \& \quad \sum \mu_0(E_i) < \mu^*(C) + \varepsilon.$$

Where are $\mu^*(C \cap E)$ & $\mu^*(C \cap E^c)$ going to come in? Well, the sets $E_i \cap E \in \mathcal{A}$ & $\bigcup_1^\infty E_i \cap E \supseteq C \cap E$, so

$$\mu^*(C \cap E) \leq \sum \mu_0(E_i \cap E).$$

Similarly, $\mu^*(C \cap E^c) \leq \sum \mu_0(E_i \cap E^c)$, so we have

since μ_0 is finitely additive

$$\mu^*(C \cap E) + \mu^*(C \cap E^c) \leq \sum \mu_0(E_i \cap E) + \sum \mu_0(E_i \cap E^c) = \sum [\mu_0(E_i \cap E) + \mu_0(E_i \cap E^c)] = \sum \mu_0(E_i)$$
$$\leq \mu^*(C) + \varepsilon$$

Since this is true $\forall \varepsilon > 0$, we see that $\mu^*(C) \geq \mu^*(C \cap E) + \mu^*(C \cap E^c)$, as desired.

Finally, we need to see that $\mu^*(E) = \mu_0(E)$. Obviously, $E \subseteq E \cup \emptyset \cup \emptyset \cup \ldots$, so $\mu^*(E) \leq \mu_0(E) + \mu_0(\emptyset) + \ldots = \mu_0(E)$.

OTOH, to show $\mu_0(E) \leq \mu^*(E)$, it suffices to show $\mu_0(E) \leq \sum \mu_0(E_i)$ for all coverings $\bigcup_1^\infty E_i \supseteq E$ w/ $E_i \in \mathcal{A}$.

We can assume the E_i are disjoint: if not, replace E_i by $E_i \setminus \bigcup_1^{i-1} E_j$.

Then, since μ_0 is a premeasure, $\sum \mu_0(E_i) = \mu_0(\bigcup_1^\infty E_i) \geq \mu_0(E)$ since $\bigcup_1^\infty E_i \supseteq E$. ∎