Math 617: Day 51

Lévy Continuity Thm: Let $X_1, X_2, \ldots$ be a seq. of real-valued r.v.'s & let X be a real r.v. Then TFAE:

① F_{X_n} converges pointwise to F_X

② X_n converges in distribution to X (i.e., P_{X_n} converges vaguely to P_X, meaning $\int f\,dP_{X_n} \to \int f\,dP$ $\forall$ continuous f w/ cpct support.)

The idea is then to see that the characteristic fcn of $\frac{1}{\sqrt{n}}\sum X_i$ converges to $e^{-t^2/2}$.

Proof: ② $\Rightarrow$ ① is obvious since $x \mapsto e^{itx}$ is bdd & continuous, so can be approximated by cont. fcts w/ cpct support.

① $\Rightarrow$ ② We need to show $E(\varphi(X_n)) \to E(\varphi(X))$ for any continuous φ w/ cpct support.

Now, we can approximate such φ by Schwartz fcts, so we may as well assume φ is Schwartz. But then

$$\varphi(X_n) = \int_{\mathbb{R}} \hat{\varphi}(t)\, e^{itX_n}\, dt$$

where $\hat{\varphi}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(x)\, e^{-itx}\, dx$ is Schwartz & hence in L^1. But then

$$E(\varphi(X_n)) = \int_{\mathbb{R}} \varphi(\xi)\, dP_{X_n}(\xi) = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \hat{\varphi}(t)\, e^{it\xi}\, dt\right) dP_{X_n}(\xi)$$

$$= \int_{\mathbb{R}} \hat{\varphi}(t)\left(\int_{\mathbb{R}} e^{it\xi}\, dP_{X_n}(\xi)\right) dt$$

$$= \int_{\mathbb{R}} \hat{\varphi}(t)\, F_{X_n}(t)\, dt$$

&, similarly, $E(\varphi(X)) = \int_{\mathbb{R}} \hat{\varphi}(t)\, F_X(t)\, dt$.

But now obviously $\hat{\varphi}(t) F_{X_n}(t) \to \hat{\varphi}(t) F_X(t)$ pointwise & $|\hat{\varphi}(t) F_{X_n}(t)| \le |\hat{\varphi}(t)|$ since $|F_{X_n}(t)| \le 1$ $\forall t$,

so Dominated Convergence implies $E(\varphi(X_n)) \to E(\varphi(X))$. ▨

In fact, this is actually something we already stated. Notice that the Fourier transform of P_X is basically a slightly different convention than when I first defined it):

$$\hat{P}_X(t) = \int e^{-i\xi t}\, dP_X(\xi) = E(e^{-itX}) = F_X(-t).$$

We stated a proposition that said if $\mu, \mu_1, \mu_2, \ldots \in M(\mathbb{R})$ w/ $\|\mu_k\| \le C < \infty$ $\forall k$ & $\hat{\mu}_k \to \hat{\mu}$ pointwise, then $\mu_k \to \mu$ vaguely.

Since $\|P_X\| = 1$ for any probability measure, this is just ① $\Rightarrow$ ② in Lévy continuity.

Now we're almost done. Let $Z_n = \frac{1}{\sqrt{n}}(X_1 + \ldots + X_n)$. The goal is to show Z_n converges to the std. Gaussian in distribution,

& the point of the above is that this will be true if $F_{Z_n} \to e^{-t^2/2}$, the characteristic function of the std. Gaussian.

So now consider F_{Z_n} or, equivalently, $\widehat{P_{Z_n}}$:

$$P_{Z_n} = P_{\frac{1}{\sqrt{n}}(X_1+\dots+X_n)} = P_{\frac{X_1}{\sqrt{n}}+\dots+\frac{X_n}{\sqrt{n}}} \overset{\text{See Day 45}}{=} P_{\frac{X_1}{\sqrt{n}}} * \dots * P_{\frac{X_n}{\sqrt{n}}}$$

So

$$F_{Z_n}(-t) = \widehat{P}_{Z_n}(t) = \widehat{P_{\frac{X_1}{\sqrt{n}}} * \dots * P_{\frac{X_n}{\sqrt{n}}}}(t) = \widehat{P}_{\frac{X_1}{\sqrt{n}}} \cdots \widehat{P}_{\frac{X_n}{\sqrt{n}}}(t) = F_{\frac{X_1}{\sqrt{n}}}(-t) \cdots F_{\frac{X_n}{\sqrt{n}}}(-t) = F_{\frac{X}{\sqrt{n}}}(-t)^n.$$

Now, $F_{\frac{X}{\sqrt{n}}}(-t) = \int e^{-it\xi/\sqrt{n}}\, dP_X(\xi) = \int e^{-i t/\sqrt{n}\, \xi}\, dP_X(\xi) = F_X(-\frac{t}{\sqrt{n}})$, so $F_{Z_n}(-t) = F_X(-t/\sqrt{n})^n$, which of

course means $F_{Z_n}(t) = F_X(t/\sqrt{n})^n$ as well.

But now since X has mean 0 & variance 1, the Taylor approximation is

$$F_X(t/\sqrt{n}) = 1 - \frac{(t/\sqrt{n})^2}{2} + o\left(\left|\tfrac{t}{\sqrt{n}}\right|^2\right) = 1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right),$$

so

$$F_{Z_n}(t) = F_X(t/\sqrt{n})^n = \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n \longrightarrow e^{-t^2/2}$$ pointwise as $n \to \infty$, and we're done.