Math 417: Day 50

Suppose $X_1, X_2, \ldots$ are i.i.d. real-valued random variables w/ mean 0 & variance σ^2. Then $\frac{1}{n}\sum_{i=1}^{n} X_i$ has mean 0 & variance σ^2/n, so it is very likely to be near 0 when n is large (of course, this is what the weak law of large numbers says).

OTOH, as we've seen, $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$ has mean 0 & variance σ^2... the content of the central limit theorem is that this goes to the Gaussian w/ mean 0 & variance σ^2.

Central Limit Theorem: Let $(X_i)_{i=1}^{\infty}$ be a seq. of i.i.d., real-valued, L^2 random variables w/ mean μ & variance σ^2. As $n \to \infty$, the distribution of $\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu)$ converges vaguely to the standard normal distribution ν_0^1.

Here the distribution $\nu_\mu^{\sigma^2}$ of the Gaussian w/ mean μ & variance σ^2 is given by

$$\nu_\mu^{\sigma^2}(E) = \int_E \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\, dt$$

& one can easily check that $\int_{\mathbb{R}} t \, d\nu_\mu^{\sigma^2}(t) = \mu$ & $\int_{\mathbb{R}} (t-\mu)^2 \, d\nu_\mu^{\sigma^2}(t) = \sigma^2$.

Now, in the CLT, we can replace X_i by $\frac{X_i - \mu}{\sigma}$, so we reduce to the case when $\mu = 0$ & $\sigma^2 = 1$.

The strategy is to use Fourier analysis, so, for any real-valued r.v. X, we define the **characteristic function** $F_X : \mathbb{R} \to \mathbb{C}$ by

$$F_X(t) = E(e^{itX}).$$

Prop: The Gaussian $\nu_\mu^{\sigma^2}$ has characteristic function $F(t) = e^{it\mu} e^{-\sigma^2 t^2/2}$.

Pf: $E(e^{itX}) = \int e^{it\xi} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} \, d\xi = e^{it\mu - \frac{t^2\sigma^2}{2}}$. ∎

If we Taylor expand e^{itX} & formally interchange expectations & sums, we expect something like $F_X(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E(X^k)$ to be true. Of course we can't really do this in general, but something more or less like this is true:

Prop: If X is a real r.v. w/ finite k^{th} moment, then F_X is C^k w/

$$\frac{d^j}{dt^j} F_X(t) = i^j E(X^j e^{itX})$$

$\forall\, 0 \le j \le k$. In particular,

$$F_X(t) = \sum_{j=0}^{k} \frac{(it)^j}{j!} E(X^j) + o(|t|^k).$$

Pf: One can check that F_X is uniformly continuous. That means we can differentiate under the integral:

$$\frac{d^j}{dt^j} F_X(t) = \frac{d^j}{dt^j} E\left(e^{itX}\right) = \frac{d^j}{dt^j} \int e^{it\xi} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi = \int \frac{d^j}{dt^j}\left(e^{it\xi}\right) dP_X(\xi)$$

$$= \int (i\xi)^j e^{it\xi} \, dP_X(\xi)$$

$$= i^j \int \xi^j e^{it\xi} \, dP_X(\xi)$$

$$= i^j E\left(X^j e^{itX}\right).$$

Then the rest follows from the usual Taylor approximation thm. ∎