Math 617: Day 5

The Jordan content is closely related to the Riemann integral: we will come back to this later, but in short we can think of

a Riemann sum as approximating a function by piecewise constant functions & then taking the sum over all values of the function of the value times the Jordan content of the set on which it takes that value.

Of course, one of the big problems of the Riemann integral is the following: we can have a uniformly bounded sequence $\{f_n\}$ of Riemann integrable functions $f_n : [0,1] \to \mathbb{R}$ which converge pointwise to $f : [0,1] \to \mathbb{R}$ which is not Riemann integrable.

For example, if $\{r_1, r_2, \dots\}$ is an enumeration of $\mathbb{Q} \cap [0,1]$ & we define $f_n = \begin{cases} 1 & \text{if } x \in \{r_1, \dots, r_n\} \\ 0 & \text{else} \end{cases}$, then f_n are Riemann integrable (since they only have finitely many discontinuities) & converge pointwise to the Dirichlet function $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$

but the Dirichlet function is totally discontinuous, & hence cannot be Riemann integrable.

As we will see, this problem goes away when we move to the Lebesgue integral.

Relatedly, Jordan measurability is not closed under taking countable unions: we can have Jordan measurable $E_1, E_2, \dots$ s.t. $\bigcup_{n=1}^{\infty} E_n$ is not Jordan measurable.

For example, singleton sets $\{x\} = [x,x]$ are elementary & hence Jordan measurable, so $\mathbb{Q} \cap [0,1]$ is a countable union of Jordan measurable sets,

but is not itself Jordan measurable. Since $\mathbb{Q} \cap [0,1]$ contains no nontrivial intervals, $m_*(\mathbb{Q} \cap [0,1]) = 0$.

On the other hand $m^*(\mathbb{Q} \cap [0,1]) = 1$ by the following lemma:

Lemma: For $E \subseteq \mathbb{R}$: (1) $m^*(E) = m^*(\overline{E})$

(2) $m_*(E) = m_*(E^\circ)$

To get countable additivity, we can modify as follows: the def of outer Jordan content is equivalent to

$$m^*(E) = \inf_{I_1 \cup \dots \cup I_n \supseteq E,\ I_1, I_2, \dots \text{ intervals}} |I_1| + \dots + |I_n|$$

But now define the Lebesgue outer measure of E to be

$$\mu^*(E) := \inf_{\bigcup_{i=1}^{\infty} I_i \supseteq E,\ I_1, I_2, \dots \text{ intervals}} \sum_{i=1}^{\infty} |I_i|$$

Ex: $\mu^*(\mathbb{Q} \cap [0,1]) = 0$. More generally, if $X \subseteq \mathbb{R}$ is countable, $\mu^*(X) = 0$ since, if $X = \{x_1, x_2, \dots\}$, then letting $I_i = [x_i, x_i] = \{x_i\}$

we have $\bigcup_{i=1}^{\infty} I_i \supseteq X$ & $\sum_{i=1}^{\infty} |I_i| = \sum_{i=1}^{\infty} 0 = 0$.

The Lebesgue outer measure is an example of an outer measure:

Def: An outer measure on a nonempty set X is a map $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ s.t.

(i) $\mu^*(\emptyset) = 0$ (ii) $\mu^*(A) \le \mu^*(B)$ if $A \subseteq B$ (iii) $\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu^*(E_i)$.

In general, we can build outer measures from collections of "elementary sets" using the following proposition:

Prop: Let $\mathcal{E} \subseteq \mathcal{P}(X)$ & $m: \mathcal{E} \to [0,\infty]$ s.t. $\emptyset, X \in \mathcal{E}$, $m(\emptyset) = 0$. Then for any $A \subseteq X$ define

$$\mu^*(A) = \inf\left\{ \sum_{i=1}^{\infty} m(E_i) : E_i \in \mathcal{E} \ \& \ \bigcup_{i=1}^{\infty} E_i \supseteq A \right\}.$$

Then μ^* is an outer measure on X.

Def: A set $A \subseteq X$ is μ^*-measurable (sometimes Carathéodory measurable) if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ $\forall$ $E \subseteq X$.

Ex: If $\mu^*(A) = 0$, then A is μ^*-measurable: $\forall$ $E \subseteq X$, $E \cap A \subseteq A$, so $\mu^*(E \cap A) = 0$. But then $E = (E \cap A) \cup (E \cap A^c)$, so

$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E \cap A^c) \leq \mu^*(E)$ since $E \cap A^c \subseteq E$. Therefore, the inequalities are equalities.

Big Important Theorem:

Carathéodory's Thm: If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra & the restriction of μ^* to $\mathcal{M}$ is a complete measure.