Math 617: Day 47

Def: A family (X_α) of random variables is *jointly independent* if the distribution of (X_α) is the product measure of the distributions of the individual X_α.

Prop: The family (X_α) is jointly independent $\iff$ for any collection $\alpha_1,\dots,\alpha_k$ of distinct indices & all measurable $E_i \subseteq R_{\alpha_i}$,

$$P(X_{\alpha_i}\in E_i \ \ \forall\, i=1,\dots,k) = \prod_{i=1}^{k} P(X_{\alpha_i}\in E_i).$$

Proof: If the (X_α) are independent, then

$$P(X_{\alpha_i}\in E_i \ \forall i) = P(X_{\alpha_1}^{-1}(E_1)\cap\dots\cap X_{\alpha_k}^{-1}(E_k)) = P((X_1,\dots,X_k)^{-1}(E_1,\dots,E_k)) = P_{(X_1,\dots,X_k)}(E_1,\dots,E_k)$$

$$= \prod_{i=1}^{k} P_{X_{\alpha_i}}(E_i) = \prod_{i=1}^{k} P(X_{\alpha_i}\in E_i)$$

Conversely, if $P(X_{\alpha_i}\in E_i \ \forall i) = \prod_{i=1}^{k} P(X_{\alpha_i}\in E_i)$, then for any collection $\{E_\alpha\}$ where all but finitely many of the E_α are equal to the corresponding R_α, we have

$$P_{(X_\alpha)}\Big(\prod_{\alpha\in A} E_\alpha\Big) = P(X_\alpha\in E_\alpha \ \forall \alpha) = P(X_{\alpha_i}\in E_i) = \prod_{i=1}^{k} P(X_{\alpha_i}\in E_i) = \prod_{\alpha\in A} P(X_\alpha\in E_\alpha)$$

since $P(X_\alpha\in R_\alpha)=1$, so $P_{(X_\alpha)}$ is the product distribution & the (X_α) are jointly independent. ∎

Ex: Let $V\subseteq \mathbb{F}_2^3$ be the subspace w/ $x_1+x_2+x_3=0$. Let (X_1,X_2,X_3) be chosen uniformly on V. Then any two of X_1,X_2,X_3 are independent, but (X_1,X_2,X_3) are not jointly independent. For example, X_3 is independent of X_1, but not of (X_1,X_2) (since $X_3=X_1+X_2$).

Prop: Let $\{X_1,\dots,X_n\}$ be jointly independent real-valued random variables. Then

$$P_{X_1+\dots+X_n} = P_{X_1} * P_{X_2} * \dots * P_{X_n}.$$

Proof: Let $S(t_1,\dots,t_n)=t_1+\dots+t_n$. Then $X_1+\dots+X_n = S(X_1,\dots,X_n)$ & so

$$P_{X_1+\dots+X_n} = S_* P_{(X_1,\dots,X_n)} = S_*\Big(\prod_{i=1}^{n} P_{X_i}\Big).$$

Hence, for any measurable $E \subseteq \mathbb{R}$,

$$P_{X_1+\dots+X_n}(E) = \left(S_*\left(\prod_{i=1}^{n} P_{X_i}\right)\right)(E) = \int_E d\left(S_*\left(\prod P_{X_i}\right)\right) = \int_{\mathbb{R}} \chi_E(t_1,\dots,t_n)\, d\left(S_*\left(\prod P_{X_i}\right)\right)$$

$$= \int \chi_E \circ S(t_1,\dots,t_n)\, d\left(\prod P_{X_i}\right)$$

Fubini

$$= \int \cdots \int \chi_E(t_1+\dots+t_n)\, dP_{X_1}(t_1)\cdots dP_{X_n}(t_n)$$

$$= P_{X_1} * \dots * P_{X_n}(E)$$

by inductively applying the definition $\mu * \nu(E) := \iint \chi_E(x+y)\, d\mu(x)\, d\nu(y)$.

Since this is true for all measurable E, we conclude that $P_{X_1+\dots+X_n} = P_{X_1} * \dots * P_{X_n}$. ☒

Note that, in particular, if the X_i are all absolutely continuous, meaning that $P_{X_i} = m_{f_i}$, then the density of $P_{X_1+\dots+X_n}$ is simply $f_1 * \dots * f_n$.

Ex: If X_i is uniformly distributed on $[-1,1]$, then $f_i = \frac{1}{2}\chi_{[-1,1]}$.

Then the density of $P_{X_1+\dots+X_n}$ is $f_1 * \dots * f_n$.

Now, $\hat{f}_i(k) = \int_{-\infty}^{\infty} \frac{1}{2}\chi_{[-1,1]}(t) e^{-ikt}\, dt = \frac{\sin(k)}{k} = \operatorname{sinc}(k)$, so the Fourier series for f_i is $\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \operatorname{sinc}(k)\, e^{ikx}$,

so $f_1 * \dots * f_n(x) = \frac{1}{(2\pi)^n}\sum_{k\in\mathbb{Z}} \operatorname{sinc}^n(k)\, e^{ikx}$.