Math 617: Day 45

Since it will come in handy soon, let's talk about Fourier transforms of measures...

First of all, if $M(\mathbb{R}^n)$ is the collection of all complex Borel measures on $\mathbb{R}^n$, then we can embed $L^1(\mathbb{R}^n)$ into $M(\mathbb{R}^n)$ by

mapping $f \in L^1(\mathbb{R}^n)$ to $\mu = m_f$, where m is Lebesgue measure.

Since we know about convolution of L^1 functions, Fourier transforms of L^1 functions, etc. more, it's useful to think about whether these

concepts extend to $M(\mathbb{R}^n)$. Indeed they do. For example...

Def: If $\mu, \nu \in M(\mathbb{R}^n)$, define the **convolution** $\mu * \nu \in M(\mathbb{R}^n)$ by

$$(\mu * \nu)(E) = \iint \chi_E(x+y)\, d\mu(x)\, d\nu(y)$$

Prop: ① Convolution of measures is commutative & associative

② For any bounded measurable function h,

$$\int h\, d(\mu * \nu) = \iint h(x+y)\, d\mu(x)\, d\nu(y)$$

③ $\|\mu * \nu\| \le \|\mu\|\,\|\nu\|$, where, for any measure space $(X, \mathcal{M}, \eta)$, the **measure norm** $\|\eta\|$ is defined by $\|\eta\| := |\eta|(X)$

④ If $\mu = m_f$ & $\nu = m_g$, then $\mu * \nu = m_{f*g}$

Proof of ④: If h is bounded & measurable, then

$$\int h\, d(\mu*\nu) \overset{③}{=} \iint h(x+y)\, d\mu(x)\, d\nu(y) = \iint h(x+y)\, f(x)\, g(y)\, dx\, dy \overset{x \mapsto x-y}{=} \iint h(x)\, f(x-y)\, g(y)\, dx\, dy = \int h(x)\, (f*g)(x)\, dx.$$ ∎

Prop: If $1 \le p \le \infty$ & $f \in L^p$, $\mu \in M(\mathbb{R}^n)$, then $(f*\mu)(x) := \int f(x-y)\, d\mu(y)$ exists for a.e. $x \in \mathbb{R}^n$ & $f*\mu \in L^p$ w/

$$\|f*\mu\|_p \le \|f\|_p\, \|\mu\|$$

Note that if $p=1$ & we identify f w/ m_f, then

$$(m_f * \mu)(E) = \iint \chi_E(x+y)\, dm_f(x)\, d\mu(y) = \iint \chi_E(x+y)\, f(x)\, dm(x)\, d\mu(y) \overset{x \mapsto x-y}{=} \iint \chi_E(x)\, f(x-y)\, dm(x)\, d\mu(y)$$

$$= \int_E \left(\int f(x-y)\, d\mu(y) \right) dm(x) = \int_E (f*\mu)(x)\, dx = m_{f*\mu}(E),$$

which is in the copy of $L^1(\mathbb{R}^n)$ inside $M(\mathbb{R}^n)$. Algebraically, this means that $L^1(\mathbb{R}^n)$ is not just a (convolution) subalgebra of $M(\mathbb{R}^n)$, but an ideal of $M(\mathbb{R}^n)$.

Anyway, the point is to extend the Fourier transform from L^1 to $M(\mathbb{R}^n)$ in the obvious way:

if $f \in L^1(\mathbb{R}^n)$, then $\hat{f}(\vec{\xi}) = \int f(x) e^{-2\pi i \vec{\xi}\cdot\vec{x}} \, dm(\vec{x}) = \int e^{-2\pi i \vec{\xi}\cdot\vec{x}} \, dm_f(\vec{x})$

So, for arbitrary $\mu \in M(\mathbb{R}^n)$, define the Fourier transform $\hat{\mu} : \mathbb{R}^n \to \mathbb{C}$ by:

$$\hat{\mu}(\vec{\xi}) := \int e^{-2\pi i \vec{\xi}\cdot\vec{x}} \, d\mu(\vec{x}).$$

Then $\hat{\mu}$ is bounded & continuous &, by ② from the Proposition, $\widehat{\mu * \nu} = \hat{\mu}\hat{\nu}$.

Def: A sequence (μ_k) of (complex) measures converges vaguely to a (complex) measure μ if

$$\int f \, d\mu_k \to \int f \, d\mu$$

for all bounded continuous functions f (this is also called convergence in the weak* topology or weak* convergence).

Prop: Suppose $\mu, \mu_1, \mu_2, \ldots \in M(\mathbb{R}^n)$ & that $\|\mu_k\| \le C < \infty$ $\forall k$ & $\hat{\mu}_k \to \hat{\mu}$ pointwise. Then $\mu_k \to \mu$ vaguely.

As a partial converse, if $\mu_k \to \mu$ vaguely & $\|\mu_k\| \to \|\mu\|$, then $\hat{\mu}_k \to \hat{\mu}$ pointwise.