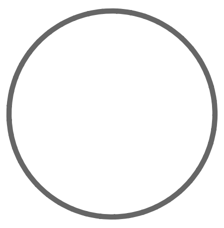
Math 617: Day 42

Application of Fourier Series: Heat diffusion on a wire loop

Fourier's original motivation for considering Fourier series was to understand heat flow in metals.

So consider a wire loop  w/ initial temperature distribution given by f. Of course, we expect that at equilibrium the temperature will equalize across the loop, yielding a uniform distribution, but what about in the interim?

Let $u(x,t)$ be the temp. at time t at location $x \in [0,1]$ on the loop. Then obviously $u(\cdot,t)$ is a function on the circle for each t (or a periodic function on $\mathbb{R}$) & $u(x,0)=f(x)$ $\forall x$.

Since $u(\cdot,t)$ is a function on the circle, it has a Fourier series:

$$u(x,t)=\sum_{n\in\mathbb{Z}} c_n(t)\, e^{2\pi i n x}$$

where $c_n(t)=\int_{S^1} u(x,t)\,e^{-2\pi i n x}\,dm(x)$ is the n^{th} Fourier coefficient of $u(\cdot,t)$.

Now, the heat equation says that $\alpha^2 u_{xx}=u_t$, where α^2 is the thermal diffusivity constant, which we'll set to be $\alpha^2=1/2$.

So now the idea is to get an ODE for $c_n(t)$. So differentiate under the integral (everything should be fairly smooth, so we'll assume this is okay):

$$c_n'(t)=\frac{d}{dt}\int_{S^1} u(x,t)\,e^{-2\pi i n x}\,dx=\int_0^1 u_t(x,t)\,e^{-2\pi i n x}\,dx.$$

Now sub in $u_t=\frac{1}{2}u_{xx}$:

integrate by parts b/c u & $e^{-2\pi i n x}$ periodic

$$c_n'(t)=\int_0^1 \tfrac{1}{2}u_{xx}(x,t)\,e^{-2\pi i n x}\,dx=\cancel{\left[\tfrac{1}{2}u_x(x,t)\tfrac{d}{dx}\left(e^{-2\pi i n x}\right)\right]_0^1}-\int_0^1 \tfrac{1}{2}u_x(x,t)\tfrac{d}{dx}\left(e^{-2\pi i n x}\right)dx$$

integrate by parts

$$=\overset{0}{\cancel{-\left[\tfrac{1}{2}u(x,t)\tfrac{d^2}{dx^2}\left(e^{-2\pi i n x}\right)\right]_0^1}}+\int_0^1 \tfrac{1}{2}u(x,t)\tfrac{d^2}{dx^2}\left(e^{-2\pi i n x}\right)dx$$

$$=\int_0^1 \tfrac{1}{2}u(x,t)\left(4\pi^2 i^2 n^2\right)e^{-2\pi i n x}\,dx$$

$$=-2\pi^2 n^2\int_0^1 u(x,t)\,e^{-2\pi i n x}\,dx$$

$$=-2\pi^2 n^2 c_n(t).$$

But now we know how to solve $c_n'(t)=-2\pi^2 n^2 c_n(t)$: the solution is just $c_n(t)=c_n(0)\,e^{-2\pi^2 n^2 t}$.

Moreover,

$$c_n(0) = \int_0^1 u(x,0)\, e^{-2\pi i n x}\, dx = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx = \hat{f}(n),$$

so the temperature at time t in location x on the wire loop is simply

$$u(x,t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{-2\pi^2 n^2 t}\, e^{2\pi i n x} = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi n(ix - \pi n t)}.$$

Notice, in particular, that for $t=0$ we have

$$u(x,0) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x} = f(x)$$

& that as $t\to\infty$, $e^{2\pi n(ix-\pi n t)} \to 0$ unless $n=0$, so

$$\lim_{t\to\infty} u(x,t) = \hat{f}(0) = \int_{S^1} f(x)\, dx$$

& the temperature equilibrates to the average of the initial temperature distribution.