# Math 617: Day 4

The usual way to construct useful measures is via the theory of outer measures. To motivate this, consider the example $X=\mathbb{R}$.

The basic idea is that, given $E\subseteq\mathbb{R}$, we should approximate $E$ by sets we know the length of (namely intervals), & then define the measure of $E$ to be the inf of the length of unions of intervals containing $E$. Of course, this doesn't always exactly work, but that motivates the following:

**Def**: The length $|I|$ of an interval $I=[a,b]$ or $[a,b)$ or $(a,b]$ or $(a,b)$ is $|I|:=b-a$.

An elementary set in $\mathbb{R}$ is a set which is a finite union of intervals. We denote the collection of elementary sets by $\mathcal{E}(\mathbb{R})$.

**Lemma**: If $E,F\subseteq\mathbb{R}$, then $E\cup F$, $E\cap F$, $E\setminus F$, and $E\Delta F=(E\setminus F)\cup(F\setminus E)$ are all elementary.

**Lemma**: An elementary set $E$ can be expressed as a union $E=I_1\cup I_2\cup\dots\cup I_n$ of intervals & the quantity

$$m(E):=|I_1|+\dots+|I_n|$$ is independent of the choice of disjoint intervals $I_1,\dots,I_n$.

**Ex**: $m\big((9,12)\cup[17,32)\big)=18$.

$m$ is often called the elementary measure of $E$, & note that it satisfies $m(\emptyset)=m((a,a))=a-a=0$ & finite additivity. It also satisfies monotonicity, since for $E\subseteq F$ elementary, $F\setminus E$ is elementary & $m(F)=m(E)+m(F\setminus E)\geq m(E)$, and finite subadditivity, since $m(E\cup F)=m(E)+m(F\setminus E)\leq m(E)+m(F)$.

Also, elementary measure satisfies translation invariance: if $x\in\mathbb{R}$ & $E+x=\{y+x : y\in E\}$, then $E+x$ is elementary & $m(E+x)=m(E)$.

**Prop**: Let $m':\mathcal{E}(\mathbb{R})\to[0,+\infty]$ be a map satisfying finite-additivity & translation invariance. Then $\exists\ c\in[0,+\infty)$ s.t. $m'(E)=cm(E)\ \forall E\in\mathcal{E}(\mathbb{R})$.

In particular, if we require $m'([0,1])=1$, then $m'\equiv m$.

**Proof**: Let $c=m'([0,1])$. Then, for $n\in\mathbb{N}$, translation invariance implies $m'([0,\frac{1}{n}))=\frac{c}{n}$ & hence $m'([a,b])=m'((0,b-a))=c(b-a)$ by rational approximation of $b-a$. Since $m'=cm$ on intervals, we must have $m'=cm$ on all elementary sets. ∎

Now, the idea is to approximate $E\subseteq\mathbb{R}$ by elementary sets $A$ & $B$: $A\subseteq E\subseteq B$.

**Def**: Let $E\subseteq\mathbb{R}$ be bounded.

The inner Jordan content $m_*(E):=\sup_{A\subseteq E,\,A\text{ elementary}} m(A)$

The outer Jordan content $m^*(E):=\inf_{B\supseteq E,\,B\text{ elementary}} m(B)$

If $m_*(E)=m^*(E)$, then $E$ is Jordan measurable & we call $m(E):=m_*(E)=m^*(E)$ the Jordan content of $E$.

**Remark:** The whole theory of elementary sets & Jordan measurability easily generalizes to $\mathbb{R}^d$, where the fundamental objects are boxes $[a_1,b_1]\times[a_2,b_2]\times\dots\times[a_d,b_d]$ whose volume is the product of the lengths of the factor intervals.

**Prop:** Let $E,F\subseteq\mathbb{R}$ be Jordan measurable:

1. (Boolean closure) $E\cup F$, $E\cap F$, $E\setminus F$ & $E\Delta F$ are Jordan measurable
2. (Non-negativity) $m(E)\geq 0$
3. (Finite additivity) If $E\cap F=\emptyset$, then $m(E\cup F)=m(E)+m(F)$
4. (Monotonicity) If $E\subseteq F$, then $m(E)\leq m(F)$.
5. (Finite subadditivity) $m(E\cup F)\leq m(E)+m(F)$
6. (Translation invariance) For $x\in\mathbb{R}$, $E+x$ is Jordan measurable & $m(E+x)=m(E)$.

**Proof of (4):** If $E\subseteq F$ & $A\subseteq E$ is elementary, then obviously $A\subseteq F$, so $m_*(E)=\sup_{A\subseteq E \text{ elem}} m(A)\leq \sup_{A\subseteq F \text{ elem}} m(A)=m_*(F)$, so

$m(E)=m_*(E)\leq m_*(F)=m(F)$. ∎