Math 617: Day 39

Prop: If $f \in \mathcal{L}^p(X, \mathcal{M}, \mu)$ for some $0 < p < \infty$, then $\mathrm{supp}(f) = \{x \in X : f(x) \neq 0\}$ is σ-finite.

Cor: If $(c_\alpha)_{\alpha \in A} \in \ell^p(A)$, then at most finitely many of the c_α are nonzero.

In particular, if $(e_\alpha)_{\alpha \in A}$ is an ON system in some Hilbert space H, then $\sum c_\alpha e_\alpha$ is well-defined & the same argument as before gives an isometric map $\ell^p(A) \to H$ given by $(c_\alpha)_{\alpha \in A} \mapsto \sum c_\alpha e_\alpha$.

Prop: Let $(e_\alpha)_{\alpha \in A}$ be an ON system in H. TFAE:

① The Hilbert space span of $(e_\alpha)_{\alpha \in A}$ is all of H.

② The algebraic span (all finite linear combinations) of $(e_\alpha)_{\alpha \in A}$ is dense in H.

③ The Parseval identity $\|x\|^2 = \sum_{\alpha \in A} |\langle x, e_\alpha \rangle|^2$ holds for all $x \in H$.

④ We have the inversion formula $x = \sum \langle x, e_\alpha \rangle e_\alpha$ $\forall x \in H$ (in particular, the $\langle x, e_\alpha \rangle$ are square summable).

⑤ The only vector orthogonal to all the e_α is the zero vector.

⑥ There is an isomorphism $\ell^2(A) \to H$ mapping $\delta_\alpha \mapsto e_\alpha$ $\forall \alpha \in A$.

A system $(e_\alpha)_{\alpha \in A}$ satisfying any of ①-⑥ is called an orthonormal basis of H.

Proof: ④ clearly implies all the rest, & ①$\Leftrightarrow$②, ③$\Rightarrow$④, ⑥$\Rightarrow$①, ⑤$\Rightarrow$① are all fairly straightforward. We will show ①$\Rightarrow$④.

Let $x \in H$. Then ① $\Rightarrow x = \sum_{\alpha \in A} c_\alpha e_\alpha$ for some $(c_\alpha)_{\alpha \in A}$.

But then only countably many of the c_α can be nonzero, so in fact $x = \sum_{n=1}^{\infty} c_n e_n$ for some subcollection $(e_n)_{n=1}^{\infty}$ of the e_α.

Then x is orthogonal to all the other e_α & the countable version of this prop $\Rightarrow x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha$, so ④ is true. ▨

Prop: Every Hilbert space has an orthonormal basis.

Cor: Every Hilbert space is isomorphic to some $\ell^2(A)$.

Proof of Prop: This is a Zorn's Lemma argument. Of course $\emptyset$ is an ON system, so every Hilbert space has an ON system.

But now we can partially order ON systems by inclusion &, by taking unions, every ascending chain has a maximal element.

So Zorn's Lemma says there's a maximal ON system $(e_\alpha)_{\alpha \in A}$. There's no unit vector orthogonal to all the e_α, or we'd get a bigger ON system by adding it. But then ⑤ says $(e_\alpha)_{\alpha \in A}$ is an ON basis. ▨

Therefore, combining w/ ④ from the Proposition, the map $H \to \ell^2(A)$ given by $x \mapsto \hat{x}$ where $\hat{x}(\alpha) = \langle x, e_\alpha \rangle$ is a unitary map.

Prop: Let A & B be sets. $\ell^2(A) \cong \ell^2(B) \iff A$ & B have the same cardinality.

Cor: All ON bases of a given Hilbert space H have the same cardinality, called the *dimension* of H.

Def: A Hilbert space is called *separable* if it has a countable dense subset.

Prop: A Hilbert space is separable $\iff$ its dimension is at most countable. This means that, up to isomorphism, there is only one infinite-dim'l separable Hilbert space.
