Math 617: Day 38

Prop: Let $T: H_1 \to H_2$ be a cont. linear transformation of Hilbert spaces. $\exists!$ cont. linear transformation $T^+: H_2 \to H_1$ s.t.

$$\langle Tx, y\rangle = \langle x, T^+ y\rangle \quad \forall\ x\in H_1,\ y \in H_2.$$

T^+ is called the (Hilbert space) adjoint of T.

Pf: Fix $y\in H_2$ & define $\lambda_y: H_1\to\mathbb{C}$ by $\lambda_y(x) = \langle Tx,y\rangle$. Then λ_y is a linear functional, so, by the Riesz representation thm,

$\exists!$ $z_y \in H_1$ s.t. $\langle x, z\rangle = \lambda_y(x) = \langle Tx, y\rangle$.

Now, define $T^+: H_2\to H_1$ by $T^+(y) := z_y$.

We can check that $\lambda_{ay_1+by_2}(x) = \langle Tx, ay_1+by_2\rangle = \bar a\langle Tx,y_1\rangle + \bar b\langle Tx, y_2\rangle = \bar a\langle x, T^+y_1\rangle + \bar b \langle x, T^+ y_2\rangle$

$= \langle x, aT^+y_1 + bT^+y_2\rangle$

so T^+ is a linear functional. Uniqueness follows from the uniqueness in Riesz. ∎

Prop: Let $T: H_1\to H_2$ be a cont. linear transformation.

1. $(T^+)^+ = T$
2. T is an isometry $\iff T^+T = id_{H_1}$
3. T is an isometric isomorphism $\iff T^+T = id_{H_1}$ & $TT^+ = id_{H_2}$
4. If $S: H_2\to H_3$, then $(ST)^+ = T^+S^+$.

Prop: The projection map $\pi_V: H\to V$ from a Hilbert space to a closed subspace is the adjoint of the inclusion map $\iota_V: V\to H$.

Orthonormal bases

First, we deal w/ countable orthonormal systems.

Suppose $e_1, e_2, \dots \in H$ is a countable ON system & $c_1, c_2,\dots\in\mathbb{C}$.

Lemma 1: $\sum_{n=1}^\infty c_n e_n$ is conditionally convergent $\iff \sum c_n^2$ converges.

Pf: Converges $\iff \|\sum c_n e_n\|^2 < \infty \iff \langle \sum c_n e_n, \sum c_n e_n\rangle < \infty \iff \sum c_i c_j \langle e_i, e_j\rangle < \infty \iff \sum c_n^2 < \infty$. ∎

Lemma 2: If $\sum c_n^2$ converges, then $\sum c_n e_n$ is unconditionally convergent, meaning permuting the order of terms doesn't change the sum.

Notation: When $(X, \mathcal{M}, \mu)$ is a measure space w/ $\mathcal{M} = \mathcal{P}(X)$ & μ being counting measure, then we denote $L^p(X, \mathcal{M}, \mu)$ by $\ell^p(X)$.

Lemma 3: The map $(c_n)_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} c_n e_n$ is an isometry from the Hilbert space $\ell^2(\mathbb{N})$ to H. The image is the smallest closed subspace containing $e_1, e_2, \dots$, called the **span** of $e_1, e_2, \dots$.

Pf: Clearly, if $(c_n)_{n=1}^{\infty} \in \ell^2(\mathbb{N})$, then $\|(c_n)\|^2 = \langle (c_n), (c_n) \rangle = \sum c_n^2 = \sum c_n^2 \langle e_n, e_n \rangle = \sum_{i,j} c_i c_j \langle e_i, e_j \rangle = \langle \sum c_n e_n, \sum c_n e_n \rangle$
$= \|\sum c_n e_n\|^2$,

so this map is an isometry. ∎

Lemma 4: Let $V = \mathrm{span}(e_1, e_2, \dots)$ & let $\pi_V : H \to V$ be the orthogonal projection map. Then

$$\pi_V x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \quad \& \quad \|\pi_V(x)\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 .$$

Corollary (Bessel's Inequality): $\forall\, x \in H$, $\sum |\langle x, e_n \rangle|^2 \leq \|x\|^2$.

Pf of Lemma 4: Let $T : \ell^2(\mathbb{N}) \to V$ be the map from Lemma 3. Then $T^+ T = id$, so $T^+(\sum c_n e_n) = (c_n)$.

But now, we can consider $\iota_V \circ T : \ell^2(\mathbb{C}) \to H$, which has adjoint $(\iota_V \circ T)^+ = T^+ \circ \iota_V^+ = T^+ \circ \pi_V$ by our earlier proposition.

Then $\langle (\iota_V \circ T)(c_n), x \rangle = \langle \sum c_n e_n, x \rangle = \sum c_n \langle e_n, x \rangle$ on the one hand &

$$\langle (\iota_V \circ T)(c_n), x \rangle = \langle (c_n), T^+ \circ \pi_V(x) \rangle = \sum c_n \overline{(\pi_V x)_n} .$$

Equating the two & noting this must be true $\forall\, n$ implies $(\pi_V x)_n = \overline{\langle e_n, x \rangle} = \langle x, e_n \rangle$, so

$$\pi_V x = \sum \langle x, e_n \rangle e_n, \quad \text{as desired.}$$ ∎