Math 617: Day 37

Hilbert Spaces

Df: A *complex inner product space* $(V, \langle , \rangle)$ is a complex vector space V together w/ a sesquilinear map $\langle , \rangle : V \times V \to \mathbb{C}$ which is sesquilinear-symmetric s.t. $\langle \vec{v}, \vec{v} \rangle \geq 0$ w/ equality $\Leftrightarrow$ $\vec{v} = \vec{0}$.

(Sesquilinear means $\langle a\vec{u} + b\vec{v}, \vec{w} \rangle = a\langle \vec{u}, \vec{w} \rangle + b\langle \vec{v}, \vec{w} \rangle$ & $\langle \vec{u}, a\vec{v} + b\vec{w} \rangle = \bar{a}\langle \vec{u}, \vec{v} \rangle + \bar{b}\langle \vec{u}, \vec{w} \rangle$

Sesquilinear-symmetric means $\langle v, w \rangle = \overline{\langle w, v \rangle}$.)

Ex: $L^2(X, \mathcal{M}, \mu)$ w/ $\langle f, g \rangle := \int_X f \bar{g} \, d\mu$

Of course, an inner product induces a norm by defining $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$, so we can talk about concepts like orthogonality in inner product spaces.

Prop: Let $\{\vec{e}_\alpha\}_{\alpha \in A}$ be a collection of orthonormal vectors in $(V, \langle , \rangle)$. If $\vec{v} \in \text{span}\{\vec{e}_\alpha\}$ (meaning $\vec{v}$ is a *finite* linear combination), then

$$\vec{v} = \sum_{\alpha \in A} \langle \vec{v}, \vec{e}_\alpha \rangle \vec{e}_\alpha$$

and

$$\|\vec{v}\|^2 = \sum_{\alpha \in A} |\langle \vec{v}, \vec{e}_\alpha \rangle|^2$$

(This is the finite version of the Plancherel theorem).

Df: A *Hilbert space* is a complete inner product space; i.e., an inner product space which is also a Banach space.

Ex: $L^2(X, \mathcal{M}, \mu)$ is a Hilbert space.

In general, we have to worry about closures a lot more than in the finite dimensional case. As we've seen, it's possible to have a proper subspace which is dense.

For example, if $V \subseteq H$ is closed, then $H = V \oplus V^\perp$, but not in general if V not closed.

Of course, there's a very natural connection b/w elts. of a Hilbert space H & linear functionals on H: if $\vec{v} \in H$, define $\varphi_{\vec{v}} \in H^*$ by $\varphi_{\vec{v}}(\vec{w}) = \langle \vec{w}, \vec{v} \rangle$. Conversely

Riesz Representation Thm: Let H be a complex Hilbert space & let $\lambda : H \to \mathbb{C}$ be a continuous linear functional. Then $\exists!$ $\vec{v} \in H$ s.t. $\lambda = \varphi_{\vec{v}}$.

Of course, we've already proved this for $L^2(X, \mathcal{M}, \mu)$.

Pf: For uniqueness, suppose $\varphi_{\vec{v}} = \varphi_{\vec{v}'}$ for $\vec{v}, \vec{v}' \in V$. Then $0 = \varphi_{\vec{v}}(\vec{w}) - \varphi_{\vec{v}'}(\vec{w}) = \langle \vec{w}, \vec{v} - \vec{v}' \rangle = \varphi_{\vec{v}-\vec{v}'}(\vec{w})$ $\forall \vec{w} \in V$,

so $\varphi_{\vec{v}-\vec{v}'} = 0$. But then $0 = \varphi_{\vec{v}-\vec{v}'}(\vec{v}-\vec{v}') = \langle \vec{v}-\vec{v}', \vec{v}-\vec{v}' \rangle$, so $\vec{v}-\vec{v}' = 0 \iff \vec{v} = \vec{v}'$.

Now, assume $\lambda \neq 0$ (otherwise, $\vec{v} = \vec{0}$ works). Then $\ker(\lambda) = \{\vec{v} \in H : \lambda(\vec{v}) = \vec{0}\}$ is a proper subspace of H which is closed since λ is continuous. But then $\ker(\lambda)^\perp \neq \{0\}$, so $\exists$ $\vec{v} \in \ker(\lambda)$ w/ $\vec{v} \neq \vec{0}$, so we can normalize s.t. $\|\vec{v}\| = 1$.

Since $\vec{v} \notin \ker(\lambda)$, $\lambda(\vec{v}) \neq \vec{0}$.

Then, $\forall \vec{w} \in H$, $\lambda\left(\vec{w} - \frac{\lambda(\vec{w})}{\lambda(\vec{v})}\vec{v}\right) = 0$, so $\vec{w} - \frac{\lambda(\vec{w})}{\lambda(\vec{v})}\vec{v} \in \ker(\lambda)$.

But then, since $\vec{v} \in \ker(\lambda)^\perp$,

$$0 = \left\langle \vec{w} - \frac{\lambda(\vec{w})}{\lambda(\vec{v})}\vec{v}, \vec{v} \right\rangle = \langle \vec{w}, \vec{v} \rangle - \frac{\lambda(\vec{w})}{\lambda(\vec{v})},$$

so $\lambda(\vec{w}) = \lambda(\vec{v}) \langle \vec{w}, \vec{v} \rangle = \langle \vec{w}, \overline{\lambda(\vec{v})}\vec{v} \rangle = \varphi_{\overline{\lambda(\vec{v})}\vec{v}}(\vec{w})$, so $\lambda = \varphi_{\overline{\lambda(\vec{v})}\vec{v}}$. ∎