Math 617: Day 36

**Prop:** $T^* \in B(W^*, V^*)$ w/ $\|T^*\|_{op} \le \|T\|_{op}$. Hence, taking transposes is a linear map $B(V,W) \to B(W^*, V^*)$.

The big theorem in this story is:

**Hahn-Banach Thm:** Let $(V, \|\cdot\|_V)$ be a normed vector space & let $U \subseteq V$ be a subspace. Then any $\lambda \in U^*$ can be extended to $\tilde{\lambda} \in V^*$

w/ the same operator norm.

**Cor:** If $V$ is nontrivial, so is $V^*$.

**Pf:** Since $V$ is nontrivial, $\exists\ \vec{v} \in V$ s.t. $\vec{v} \neq \vec{0}$. Let $U = \{\lambda \vec{v} : \lambda \in \mathbb{C}\}$ be the span of $\vec{v}$ & define $\varphi : U \to \mathbb{C}$ by $\varphi(\lambda \vec{v}) = \lambda \|\vec{v}\|_V$.

Then $|\varphi \vec{u}| = |\varphi \lambda \vec{v}| = |\lambda| \|\vec{v}\|_V = \|\lambda \vec{v}\|_V = \|\vec{u}\|_V$ for all $\vec{u} \in U$, so $\|\varphi\|_{op} = 1$. But then, by Hahn-Banach, $\varphi$ extends to

a continuous linear fnl $\tilde{\varphi} : V \to \mathbb{C}$ w/ $\|\tilde{\varphi}\|_{op} = 1$; in particular, $\tilde{\varphi} \neq 0$.

We prove Hahn-Banach in stages:

**Prop:** Hahn-Banach is true when $V$ is a real vector space & $V$ is spanned by $U$ & one other vector $\vec{v}$.

**Proof:** This is trivial if $\vec{v} \in U$, so assume $\vec{v} \notin U$. Also, normalize s.t. $\|\lambda\|_{op} = 1$. Now, by hypothesis every elt. of $V$ is of the form

$\vec{u} + t\vec{v}$ for $t \in \mathbb{R}$, $\vec{u} \in U$ & $\tilde{\lambda}(\vec{u} + t\vec{v}) = \tilde{\lambda}(\vec{u}) + t\tilde{\lambda}(\vec{v}) = \lambda(\vec{u}) + t\tilde{\lambda}(\vec{v})$, so it's enough to define $\tilde{\lambda}(\vec{v})$.

Now, we need $|\tilde{\lambda}(\vec{u} + t\vec{v})| \le \|\vec{u} + t\vec{v}\|_V$ $\forall\ \vec{u} \in U$ & $t \in \mathbb{R}$. Clearly this is true when $t = 0$.

For $t \neq 0$, the inequality is still true when we replace $\vec{u}$ by $-t\vec{u}$:

$$|t|\,|\tilde{\lambda}(\vec{v}) - \lambda(\vec{u})| = |\tilde{\lambda}(-t\vec{u} + t\vec{v})| \le \|-t\vec{u} + t\vec{v}\|_V = |t|\,\|\vec{v} - \vec{u}\|_V$$

Divide both sides by $|t|$ to get

$$-\|\vec{v} - \vec{u}\| \le \tilde{\lambda}(\vec{v}) - \lambda(\vec{u}) \le \|\vec{v} - \vec{u}\|_V$$

or, equivalently,

$$\lambda(\vec{u}) - \|\vec{v} - \vec{u}\|_V \le \tilde{\lambda}(\vec{v}) \le \lambda(\vec{u}) + \|\vec{v} - \vec{u}\|_V$$

Clearly the quantity on the LHS is $\le$ the RHS, so we can choose any value for $\tilde{\lambda}(\vec{v})$ that makes this work $\forall\ \vec{u} \in U$

(for example, any number $\le 2 \inf_{\vec{u} \in U} \|\vec{v} - \vec{u}\|_V$ will do). ▨

Now, the proof of the full real Hahn-Banach thm is an application of Zorn's Lemma:

**Proof of real Hahn-Banach:** Define a partial extension of $\lambda$ to be a pair $(U', \lambda')$ s.t. $U' \supseteq U$ & $\lambda'$ is an extension of $\lambda$ to $U'$ w/ $\|\lambda'\|_{op} = \|\lambda\|_{op}$

The collection of partial extensions is partially ordered w/ the relation $(Y'',\lambda'') \geq (Y',\lambda')$ when $Y'' \supseteq Y'$ & $\lambda''$ extends $\lambda'$.

Every increasing chain has an upper bound, so Zorn's lemma says there is some maximal extension $(\vec{U},\vec{\lambda})$. If $\vec{U} = V$, we're done; if not, $\exists\, \vec{v} \in V$ w/ $\vec{v} \notin \vec{U}$ & we can apply the proposition to get an extension of $(\vec{U},\vec{\lambda})$ $(\Rightarrow\Leftarrow)$. ☒

To prove the complex version, we let $\rho = \mathrm{Re}(\lambda)$; then the real version implies $\exists$ an extension $\tilde{\rho}$ of $\rho$ to all of $V$.

But then define $\tilde{\lambda}(\vec{v}) := \tilde{\rho}(\vec{v}) - i\tilde{\rho}(i\vec{v})$, which you can check is complex-linear & has $\|\tilde{\lambda}\|_{op} = \|\lambda\|_{op}$.

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Obviously, given $V$ we can form the double dual $V^{**} = (V^*)^*$. There is a natural map $\iota : V \to V^{**}$ given by

$$\iota(\vec{v})(\lambda) = \lambda(\vec{v}) \qquad \forall\, \lambda \in V^*.$$

This is continuous, linear, & $\|\iota\|_{op} \leq 1$. In fact:

Thm: $\iota$ is an isometric isomorphism onto its image.

If $\iota$ is surjective (i.e. $V$ is isometrically isomorphic to $V^{**}$), then we say that $V$ is reflexive. Of course, $V^{**}$ is a Banach space, so only Banach spaces can be reflexive.

We've seen that finite-dim'l vector spaces are reflexive, as are the $L^p$ spaces for $1 < p < \infty$.

Thm: If $T : V \to W$ is a cont. linear transformation of normed vector spaces, then $\|T^*\|_{op} = \|T\|_{op}$, so the transpose is an isometric embedding

$$B(V,W) \to B(W^*, V^*)$$

Pf: We already know $\|T^*\|_{op} \leq \|T\|_{op}$, so we only need to show the reverse inequality.

Let $\alpha < \|T\|_{op}$. Then $\exists\, \vec{v} \in V$ s.t. $\|T\vec{v}\|_W \geq \alpha \|\vec{v}\|_V$. By the proof of the earlier corollary, $\exists\, \lambda \in W^*$ s.t. $\|\lambda\|_{W^*} = 1$.

But then $\|T^*\lambda(\vec{v})\| = \|\lambda(T\vec{v})\| = \|T\vec{v}\|_W \geq \alpha \|\vec{v}\|_V$, so $\|T^*\lambda\| \geq \alpha$ $\forall\, \alpha < \|T\|_{op}$. But then $\|T^*\|_{op} \geq \alpha \Rightarrow \|T^*\|_{op} \geq \|T\|_{op}$ ☒