Math 617: Day 35

What we're seeing is an example of duality of normed vector spaces. In particular, as we'll see shortly, the preceding theorem is telling us that, if $\frac{1}{p}+\frac{1}{q}=1$, then L^q is (isomorphic to) the dual of L^p.

More precisely, in the category of normed vector spaces, the morphisms are continuous (or bounded) linear transformations.

If $T: V \to W$ is a continuous linear transformation, we know $\exists\, C>0$ s.t. $\|T\vec{v}\|_W \leq C\|\vec{v}\|_V$ $\forall\, \vec{v}\in V$. The smallest such C is called the operator norm of T, denoted $\|T\|_{op} = C$.

Two normed vector spaces are isomorphic if $\exists$ an invertible continuous linear transformation b/w them. More precisely, if $(V,\|\cdot\|_V)$ & $(W,\|\cdot\|_W)$ are two normed vector spaces, $\exists$ invertible, continuous, linear $T: V\to W$ s.t.

$$c\|\vec{v}\|_V \leq \|T\vec{v}\|_W \leq C\|\vec{v}\|_V \quad \text{for constants } c, C>0.$$

If you can take $c=C=1$, then T is an isometric isomorphism.

Def: Let $B(V,W)$ be the space of all continuous ($\Leftrightarrow$ bounded) linear transformations from $(V,\|\cdot\|_V)$ to $(W,\|\cdot\|_W)$. This is clearly a vector space.

Prop: $B(V,W)$ w/ the operator norm is a normed vector space. If $(W,\|\cdot\|_W)$ is a Banach space, so is $B(V,W)$.

Def: Let V be a normed vector space over the field $k=\mathbb{R}$ or $\mathbb{C}$. The (continuous) dual space V^* of V is $V^* := B(V,k)$.

Note: By the previous Proposition, the dual of any normed vector space is a Banach space (which gives you some idea why duals are often nice things to think about).

Ex: $(\mathbb{R}^n)^* \cong \mathbb{R}^n$, $(\mathbb{C}^n)^* \cong \mathbb{C}^n$.

Exercise: Let $c_c(\mathbb{N})$ be the space of sequences w/ only finitely many nonzero (complex) numbers, with the norm $\|(a_n)\|_\infty = \max|a_n|$. Then $c_c(\mathbb{N})^* \cong \ell^1(\mathbb{N})$, but $\ell^1(\mathbb{N})^* = (c_c(\mathbb{N})^*)^* \cong \ell^\infty(\mathbb{N})$.

Prop: If $(X,\mathcal{M},\mu)$ is σ-finite, $1\leq p<\infty$ & $\frac{1}{p}+\frac{1}{q}=1$, then $L^p(X,\mathcal{M},\mu)^*$ is isometrically isomorphic to $L^q(X,\mathcal{M},\mu)$.

Def: Let $T\in B(V,W)$. The transpose $T^*: W^* \to V^*$ is defined by

$$T^*\lambda := \lambda\circ T \quad \text{for any } \lambda\in W^*.$$

In other words, $(T^*\lambda)(\vec{v}) = \lambda(T\vec{v})$ $\forall\, \vec{v}\in V$.