Math 617: Day 34

One of the most important applications of Hölder's inequality is to the study of linear functionals on L^p spaces.

If p & q are conjugate exponents (i.e. $\frac{1}{p}+\frac{1}{q}=1$), then for any $g\in L^q(X,\mathcal{M},\mu)$ we get a linear functional $\varphi_g : L^p(X,\mathcal{M},\mu)\to\mathbb{C}$ defined by

$$\varphi_g(f) := \int_X f\,\bar{g}\,d\mu$$

By Hölder, this is a bounded linear functional: $|\varphi_g(f)| = \left|\int_X f\bar{g}\,d\mu\right| \le \int_X |f\bar{g}|\,d\mu \le \|f\|_{L^p}\|g\|_{L^q} = C\,\|f\|_{L^p}$ (where $C=\|g\|_{L^q}$)

And then, by the following Lemma, φ_g is continuous:

Lemma: Let $(X,\|\,\|_X)$, $(Y,\|\,\|_Y)$ be normed vector spaces & let $T:X\to Y$ be linear. Then TFAE:

① T is continuous.

② T is continuous at 0.

③ $\exists$ constant C s.t. $\|Tx\|_Y \le C\|x\|_X$ $\forall x\in X$.

Pf: ①⇒② is obvious, as is ③⇒②.

For ②⇒①, let $\varepsilon>0$. By ②, $\exists\ \delta>0$ s.t. $\|x\|_X<\delta \Rightarrow \|Tx\|_Y<\varepsilon$. But then, for any $x\in X$, $\|x-x_0\|_X<\delta \Rightarrow \varepsilon > \|T(x-x_0)\| = \|Tx-Tx_0\|$
by linearity of T.

For ①⇒③, let $B=B_Y(0,1)$ be the unit ball in Y. Then continuity of T implies $T^{-1}(B)$ is open in X, so $\exists\ r>0$ s.t.

$\|x\|_X<r \Rightarrow Tx\in B \Rightarrow \|Tx\|_Y<1$. But then $\forall x\in X$, $x=\frac{2\|x\|_X}{r}\cdot\frac{r}{2\|x\|_X}x$ & $\left\|\frac{r}{2\|x\|_X}x\right\| = \frac{r}{2}<r$, so

$$\|Tx\|_Y = \left\|T\left(\frac{2\|x\|_X}{r}\cdot\frac{r}{2\|x\|_X}x\right)\right\|_Y = \frac{2\|x\|_X}{r}\left\|T\left(\frac{r}{2\|x\|_X}x\right)\right\| < \frac{2\|x\|_X}{r}\cdot 1 = \frac{2}{r}\|x\|_X$$

& ③ follows w/ $C=\frac{2}{r}$ (which is constant). ∎

So we see that the functionals φ_g give continuous linear functionals on L^p; in fact, this is essentially the only way to build continuous linear functionals on L^p:

Thm: Let $1\le p<\infty$ & assume $(X,\mathcal{M},\mu)$ is σ-finite. Let $\lambda: L^p(X,\mathcal{M},\mu)\to\mathbb{C}$ be a continuous linear functional.

Then $\exists!\ g\in L^q(X,\mathcal{M},\mu)$ s.t. $\varphi_g=\lambda$.

Pf: Uniqueness is straightforward: note that g also defines a complex measure μ_g & that $\int f\,d\mu_g = \int f\bar{g}\,d\mu$; as we've seen, g is unique up to a.e. equality.

I'll prove existence in the case μ is finite; the σ-finite case follows from writing $X=\bigcup E_n$ w/ $\mu(E_n)<\infty$.

Now, $\lambda: L^p(X,\mathcal{M},\mu)\to\mathbb{C}$ defines a function $\nu:\mathcal{M}\to\mathbb{C}$ by $\nu(E):=\lambda(\chi_E)$. Linearity of λ implies ν is finitely additive & $\nu(\emptyset)=0$.

Now, if $E_1, E_2, \ldots$ are disjoint, then $\chi_{\bigcup_{i=1}^{n} E_i} \to \chi_{\bigcup_{i=1}^{\infty} E_i}$ in L^p by dominated convergence & finiteness of μ.

But then $\nu(\bigcup_{i=1}^{\infty} E_i) = \lambda(\chi_{\bigcup_{i=1}^{\infty} E_i}) = \lim_{n\to\infty} \lambda(\chi_{\bigcup_{i=1}^{n} E_i}) = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda(\chi_{E_i}) = \lim_{n\to\infty} \sum_{i=1}^{n} \nu(E_i) = \sum_{i=1}^{\infty} \nu(E_i)$,

so ν is countably additive & hence a complex measure.

Further, if $\mu(E) = 0$, then $\nu(E) = \lambda(\chi_E) = \lambda(0) = 0$, so ν is abs. cont. w.r.t. μ.

Therefore, by the Radon-Nikodym thm, $\nu = \mu_g$ for some $g \in L^1(X, \mathcal{M}, \mu)$ & $\lambda(\chi_E) = \varphi_g(\chi_E)$ for all measurable E.

By linearity, this means $\lambda = \varphi_g$ on all simple functions. But then, taking uniform limits & then monotone limits, we get that

$\lambda = \varphi_g$ on all of $L^p(X, \mathcal{M}, \mu)$; in particular, $\int_X f\bar{g}\, d\mu = \lambda(f)$ is finite $\forall f \in L^p(X, \mathcal{M}, \mu)$.

Now, it remains to show that $g \in L^q(X, \mathcal{M}, \mu)$ w/ $\frac{1}{p} + \frac{1}{q} = 1$. By taking real & imaginary parts & then positive & negative parts,

we may as well assume $g \geq 0$.

Since $\varphi_g = \lambda$ is continuous, $\exists$ C s.t. $|\varphi_g(f)| \leq C\|f\|_{L^p}$ $\forall f \in L^p(X, \mathcal{M}, \mu)$

First, consider the case $p > 1$.

What you really want to do is now apply this in the case $f = g^{q-1}$: then we'd have

$$\|g\|_{L^q}^q = \int_X g^q\, d\mu = \int_X g^{q-1}\bar{g}\, d\mu = \int_X f\bar{g}\, d\mu = \varphi_g(f) \leq C\|f\|_{L^p} = C\|g^{q-1}\|_{L^p} = C\|g\|_{L^{p(q-1)}}^{q-1} = C\|g\|_{L^q}^{q-1}$$

since $p = \frac{q}{q-1}$. But then we can divide both sides by $\|g\|_{L^q}^{q-1}$ to get $\|g\|_{L^q} \leq C \Rightarrow g \in L^q$.

Of course, this is cheating since the above manipulations don't mean anything unless we already know $\|g\|_{L^q} < \infty$.

But this is an easy fix: let $f_N = \min(N, g)^{q-1}$ for some large N. This is in L^p since it is bdd, g is absolutely integrable, & $\mu(X) < \infty$. Then

$$\|\min(N,g)\|_{L^q}^q = \int_X \min(N,g)^q\, d\mu \leq \int_X \min(N,g)^{q-1} g\, d\mu = \varphi_g(f) \leq C\|f\|_{L^p} = C\|\min(N,g)^{q-1}\|_{L^p} = \|\min(N,g)\|_{L^q}^{q-1}$$

& so $\|\min(N,g)\|_{L^q} \leq C$. Letting $N \to \infty$ & using monotone convergence gives us $\|g\|_{L^q} \leq C < \infty$, as needed.

For $p = 1$, we have $q = \infty$, so the above doesn't work. Instead, we apply $|\varphi_g(f)| \leq C\|f\|_{L^p}$ to $f = \chi_{g>N}$ to see that

g is essentially bdd by C:

$$|\varphi_g(f)| = \left|\int_X f\bar{g}\, d\mu\right| = \int_X \chi_{g>N}\, g\, d\mu \geq \int_X \chi_{g>N} N\, d\mu = N\mu(\{x : g(x) > N\}).$$

OTOH, $|\varphi_g(f)| \leq C\|f\|_{L^1} = C\int_X \chi_{g>N}\, d\mu = C\mu(\{x : g(x) > N\})$, so $N\mu(\{x : g(x) > N\}) \leq C\mu(\{x : g(x) > N\})$ $\forall N$, which implies that $\mu(\{x : g(x) > N\}) = 0$ when $N > C$, so g is essentially bdd by $C \Rightarrow g \in L^\infty$. $\blacksquare$