Math 617: Day 33

Thm (Hölder's Inequality): Let $f \in \mathcal{L}^p(X,\mathcal{M},\mu)$ & $g \in \mathcal{L}^q(X,\mathcal{M},\mu)$ for $0 < p,q \le \infty$. Then $fg \in \mathcal{L}^r(X,\mathcal{M},\mu)$ & $\|fg\|_{\mathcal{L}^r} \le \|f\|_{\mathcal{L}^p}\|g\|_{\mathcal{L}^q}$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Cor (Cauchy-Schwarz Inequality): For $f,g \in \mathcal{L}^2(X,\mathcal{M},\mu)$,

$$\left|\int_X f(x)\overline{g(x)}\,d\mu(x)\right| \le \|f\|_{\mathcal{L}^2}\|g\|_{\mathcal{L}^2}.$$

Proof of Hölder: This is easy when $p=\infty$ or $q=\infty$, so assume $0<p,q<\infty$. Notice that

$$\|f^a\|_{\mathcal{L}^b} = \left(\int_X |f^a|^b\,d\mu\right)^{1/b} = \left(\int_X |f|^{ab}\,d\mu\right)^{1/b} = \left(\left(\int_X |f|^{ab}\,d\mu\right)^{\frac{1}{ab}}\right)^a = \|f\|_{\mathcal{L}^{ab}}^a$$

for any a & b.

Now, it suffices to show

$$\|fg\|_{\mathcal{L}^r}^r \le \|f\|_{\mathcal{L}^p}^r \|g\|_{\mathcal{L}^q}^r$$

So, letting $a=r$ & substituting in r, p & q for b, we see that it's enough to show

$$\|f^r g^r\|_{\mathcal{L}^1} \le \|f^r\|_{\mathcal{L}^{p/r}} \|g^r\|_{\mathcal{L}^{q/r}}$$

& that $\frac{1}{p/r} + \frac{1}{q/r} = \frac{r}{p} + \frac{r}{q} = r\left(\frac{1}{p}+\frac{1}{q}\right) = r\cdot\frac{1}{r} = 1$,

So it's enough to prove the theorem in the case $\frac{1}{p}+\frac{1}{q}=1$ (so called *dual exponents*).

Now, the claim is obvious if $\|f\|_{\mathcal{L}^p}=0$ or $\|g\|_{\mathcal{L}^q}=0$, so, using homogeneity, we may as well normalize to $\|f\|_{\mathcal{L}^p}=1=\|g\|_{\mathcal{L}^q}$, so then the goal is to show

$$\int_X |fg|\,d\mu \le 1.$$

Again we will use convexity: the exponential function $t \mapsto e^t$ is convex on $[0,+\infty)$. But then that means the function $H(t) := a^{1-t}b^t$ is convex for all t. Hence, $H(\lambda t_1 + (1-\lambda)t_2) \le \lambda H(t_1) + (1-\lambda)H(t_2)$ $\forall\, \lambda, t_1, t_2$.

In particular, w/ $t_1 = 1$ & $t_2 = 0$, $a^{1-\lambda}b^\lambda = H(\lambda) \le \lambda H(0) + (1-\lambda)H(1) = \lambda b + (1-\lambda)a$.

Now, let $a = |f(x)|^p$, $b = |g(x)|^q$ & $\lambda = \frac{1}{p}$ for any $x \in X$. Then

$$|f(x)g(x)| = (|f(x)|^p)^{1/p}(|g(x)|^q)^{1/q} = (|f(x)|^p)^{1-1/q}(|g(x)|^q)^{1/q} \le \frac{1}{q}|g(x)|^q + \frac{1}{p}|f(x)|^p.$$

But then integrating both sides gives $\|fg\|_{\mathcal{L}^1} \le \frac{1}{p}+\frac{1}{q} = 1 = \|f\|_{\mathcal{L}^p}\|g\|_{\mathcal{L}^q}$. ∎

Ex: Consider $f = A\chi_E$ & $g = B\chi_F$ for $A, B > 0$ & E, F measurable.

Then, by Hölder,

$$AB\mu(E\cap F)^{1/r} = \left(\int_X |A\chi_E B\chi_F|^r d\mu\right)^{1/r} = \|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q} = \left(\int_X |A\chi_E|^p d\mu\right)^{1/p}\left(\int_X |B\chi_F|^q d\mu\right)^{1/q} = A\mu(E)^{1/p} B\mu(F)^{1/q}$$

or, equivalently, $\mu(E\cap F)^{1/r} \leq \mu(E)^{1/p}\mu(F)^{1/q}$.

Of course, we could have figured this out on our own: $\mu(E\cap F) \leq \mu(E)$ & $\mu(E\cap F) \leq \mu(F)$, so

$$\mu(E\cap F)^{1/r} = \mu(E\cap F)^{1/p + 1/q} = \mu(E\cap F)^{1/p}\mu(E\cap F)^{1/q} \leq \mu(E)^{1/p}\mu(F)^{1/q}.$$

We only get equality when $\mu(E\cap F) = \mu(E) = \mu(F)$... i.e., f & g are multiples μ-a.e.

In fact, this is true in general:

Prop: Let $0 < p, q < \infty$ & $f \in L^p(X, \mathcal{M}, \mu)$, $g \in L^q(X, \mathcal{M}, \mu)$ s.t. $\|fg\|_{L^r} = \|f\|_{L^p}\|g\|_{L^q}$. Then $f^p = a g^q$ μ-a.e. for some scalar a.