Math 617: Day 32

We've now seen that $L^p(X,\mathcal{M},\mu)$ is a normed vector space & hence a metric space.

Of course, given a metric, we get a topological structure in the usual way & we can talk about convergence: a sequence $f_n\in L^p(X,\mathcal{M},\mu)$ **converges to** $f\in L^p(X,\mathcal{M},\mu)$ if $\|f_n-f\|_{L^p}\to 0$ as $n\to\infty$, & the limit is unique if it exists.

Now, the point is that L^p is **complete** with respect to this notion of convergence, which is to say:

Prop: $L^p(X,\mathcal{M},\mu)$ is a **Banach space** for $1\le p\le\infty$.

Proof: We use the basic fact that a normed vector space is complete if & only if all absolutely convergent series are actually convergent.

In particular, it suffices to show that any series $\sum_{n=1}^{\infty} f_n$ of functions in L^p which is absolutely convergent (i.e., $\sum_{n=1}^{\infty}\|f_n\|_{L^p}<\infty$) actually converges.

Of course, if $p=\infty$ this is trivial since a.e. pointwise sum converges absolutely. If $1\le p<\infty$, let $M=\sum\|f_n\|_{L^p}$.

Let $G_n=\sum_{k=1}^{n}|f_k|$. Then, by the triangle inequality, $\|G_n\|_{L^p}=\|\sum_{k=1}^{n}|f_k|\|_{L^p}\le\sum_{k=1}^{n}\|f_k\|\le\sum_{k=1}^{\infty}\|f_k\|=M$ for all n.

But then monotone convergence implies

$$\Big\|\sum_{n=1}^{\infty}|f_n|\Big\|_{L^p}^p=\int_X\Big(\sum_{n=1}^{\infty}|f_n|\Big)^p d\mu=\int_X\Big(\lim_{n\to\infty}\sum_{k=1}^{n}|f_n|\Big)^p d\mu=\lim_{n\to\infty}\int_X\Big(\sum_{k=1}^{n}|f_n|\Big)^p d\mu=\lim_{n\to\infty}\int_X G_n^p\,d\mu=\lim_{n\to\infty}\|G_n\|_{L^p}^p\le M^p,$$

so $G:=\sum_{n=1}^{\infty}|f_n|$ is in $L^p(X,\mathcal{M},\mu)$. But then $\sum_{n=1}^{\infty}f_n(x)$ is absolutely convergent for μ-a.e. $x\in X$.

Therefore, $\exists$ F s.t. $F(x)=\sum_{n=1}^{\infty}f_n(x)$ almost everywhere & $|F|\le G$, so $F\in L^p(X,\mathcal{M},\mu)$.

In turn, $|F-\sum_{k=1}^{n}f_k|^p\le(|F|+\sum_{k=1}^{n}|f_k|)^p\le(G+G)^p=2^pG^p\in L^1(X,\mathcal{M},\mu)$, so dominated convergence implies

$$\Big\|F-\sum_{k=1}^{n}f_k\Big\|_{L^p}^p=\int_X\Big|F-\sum_{k=1}^{n}f_k\Big|^p d\mu\to 0,$$

which says $\sum_{k=1}^{n}f_k$ converges to F in $L^p(X,\mathcal{M},\mu)$ & we're done. ∎

Prop: For $0<p<\infty$, the subspace of simple functions with finite measure support is a dense subspace in $L^p(X,\mathcal{M},\mu)$.

Proof Idea: Approximate L^p functions by bounded functions, approx. bounded functions by bounded functions w/ finite support, & approx. those by simple functions w/ finite support. ∎

Now, L^p is generally not an algebra, since we can multiply two L^p functions & get a function which isn't in L^p, but the family of L^p spaces does have the following algebraic property:

Thm (Hölder's Inequality): Let $f\in L^p(X,\mathcal{M},\mu)$ & $g\in L^q(X,\mathcal{M},\mu)$ for $0<p,q\le\infty$. Then $fg\in L^r(X,\mathcal{M},\mu)$ & $\|fg\|_{L^r}\le\|f\|_{L^p}\|g\|_{L^q}$

where $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$.