Math 617: Day 31

We now begin to discuss L^p spaces, which turn out to be vector spaces w/ algebraic, norm, metric, topological, functional and sometimes inner product structures & which arise (at least in part) as the theoretical justification behind things like Fourier analysis.

We can establish essentially all of these things for arbitrary measure spaces $(X, \mathcal{M}, \mu)$, though of course the most important examples are $\mathbb{R}^n$ or $\mathbb{C}^n$ w/ Lebesgue measure.

Df: A p^{th}-power integrable function is a function $f: X \to \mathbb{C}$ s.t. $\int_X |f|^p \, d\mu < \infty$.

The L^p space $L^p(X, \mathcal{M}, \mu)$ is the set of p^{th}-power integrable functions modulo the equivalence relation $f \sim g$ if $f = g$ μ-a.e.

Lemma: $L^p(X, \mathcal{M}, \mu)$ is a vector space.

In fact, we want to see that it is a normed vector space.

Df: For $f \in L^p(X, \mathcal{M}, \mu)$, define the p norm $\|f\|_{L^p}$ by $\|f\|_{L^p} := \left(\int_X |f|^p \, d\mu\right)^{1/p}$

Of course, this really is a norm:

Lemma: For $0 < p < \infty$ & $f, g \in L^p(X, \mathcal{M}, \mu)$,

① (Non-degeneracy) $\|f\|_{L^p} = 0 \iff f = 0$

② (Homogeneity) $\|cf\|_{L^p} = |c| \, \|f\|_{L^p}$ for all $c \in \mathbb{C}$.

③ ((Quasi-)triangle inequality) $\|f+g\|_{L^p} \le C\left(\|f\|_{L^p} + \|g\|_{L^p}\right)$ for some C depending on p. If $p \ge 1$, then $C = 1$ works.

Pf: ① & ② are trivial. The first part of ③ is an exercise; now, suppose $1 \le p < \infty$. The triangle inequality is trivially true if $f = 0$ or $g = 0$, so we can assume $\|f\|_{L^p} \neq 0$ & $\|g\|_{L^p} \neq 0$. Using homogeneity, rescale so that $\|f\|_{L^p} + \|g\|_{L^p} = 1$, which implies

$f = (1-\theta)F$ & $g = \theta G$ for $0 < \theta < 1$, $F, G \in L^p(X, \mathcal{M}, \mu)$ & $\|F\|_{L^p} = 1 = \|G\|_{L^p}$. So now the goal is to show

$$\|f+g\|_{L^p}^p = \|(1-\theta)F + \theta G\|_{L^p}^p = \int_X |(1-\theta)F + \theta G|^p \, d\mu \le 1^p = 1.$$

But now $z \mapsto |z|^p$ is convex on $\mathbb{C}$ when $1 \le p < \infty$, which means that

$$|(1-\theta)F + \theta G|^p \le (1-\theta)|F|^p + \theta |G|^p.$$

And then $\int_X |(1-\theta)F + \theta G|^p \, d\mu \le \int \left((1-\theta)|F|^p + \theta |G|^p\right) d\mu = (1-\theta)\|F\|_{L^p}^p + \theta \|G\|_{L^p}^p = 1 - \theta + \theta = 1$,

as needed. ▨

In particular, this tells us $L^p(X, \mathcal{M}, \mu)$ is a quasi-normed vector space for all $0 < p < \infty$ & a normed vector space for $1 \le p < \infty$.

(Note that it's only a seminorm if we don't mod out by the equivalence relation)

In fact, we can still get something sensible when $p \to \infty$. First the definition, then the justification of why we call it L^∞:

Def: A function $f: X \to \mathbb{C}$ is essentially bounded if $\exists\, M \geq 0$ s.t. $|f(x)| \leq M$ μ-a.e. Define $\|f\|_{L^\infty}$ to be the smallest such M.

We define $L^\infty(X, \mathcal{M}, \mu)$ to be the space of essentially bounded functions modulo equality almost everywhere w/ the $\|\cdot\|_{L^\infty}$ norm.

Ex: Suppose $f = A\chi_E$ for $A > 0$ & $E \in \mathcal{M}$. Then, assuming $\mu(E) > 0$,

$$\|f\|_{L^p} = \left(\int_X |f|^p \, d\mu\right)^{1/p} = \left(\int_X (A\chi_E)^p \, d\mu\right)^{1/p} = A\left(\int_X \chi_E \, d\mu\right)^{1/p} = A\mu(E)^{1/p}$$

for all $p > 0$ & $\lim_{p\to\infty} \|f\|_{L^p} = \lim_{p \to \infty} A\mu(E)^{1/p} = A = \|f\|_{L^\infty}$.

In fact, it is more generally true that, if $f \in L^\infty \cap L^{p_0}$ for some $p_0 > 0$, then $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$.

Of course, given a normed vector space, it is also a metric space.

Prop: If $(V, \|\cdot\|)$ is a normed vector space, then the function $d: V \times V \to [0, +\infty)$ given by $d(f,g) = \|f - g\|$ is a metric.

Moreover, d is translation-invariant (i.e. $d(f+h, g+h) = d(f,g)$ $\forall f,g \in V$) & homogeneous (i.e. $d(cf, cg) = |c|\, d(f,g)$ $\forall f,g \in V$ & all scalars c)

Conversely, all translation-invariant, homogeneous metrics on V arise from exactly one norm on V in this way.