Math 417: Day 30

In the higher-dim'l Lebesgue differentiation thm, we can replace the balls $B(\vec{x}, r)$ w/ sets that shrink nicely to x:

Def: A collection $\{E_r\}_{r>0}$ of sets shrinks nicely to x if:

- $E_r \subseteq B(\vec{x}, r)$ $\forall r$
- $m(E_r) \geq c\, m(B(\vec{0}, r))$ for some c independent of r.

Higher-Dimensional Lebesgue Differentiation Thm, v.2: Let $f: \mathbb{R}^d \to \mathbb{C}$ be locally integrable. Then for a.e. $\vec{x} \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)|\, dm(y) = 0 \quad \& \quad \lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y)\, dm(y) = f(x)$$

for every family $\{E_r\}_{r>0}$ that shrinks nicely to $\vec{x}$.

As in the one-dimensional case, the proof is a density argument. The theorem is true for continuous functions. The key quantitative estimate is:

Hardy-Littlewood maximal inequality: Let $f: \mathbb{R}^d \to \mathbb{C}$ be absolutely integrable & let $\lambda > 0$. Then

$$m\left(\left\{x \in \mathbb{R}^d : \sup_{r>0} \frac{1}{m(B(\vec{x}, r))} \int_{B(\vec{x}, r)} |f(\vec{y})|\, dm(\vec{y}) \geq \lambda\right\}\right) \leq \frac{C_d}{\lambda} \int_{\mathbb{R}^d} |f(\vec{x})|\, dm(\vec{x})$$

for some constant $C_d > 0$ depending only on d.

Remark: The optimal C_d is only known for $d = 1$, where we have $C_1 = \frac{11 + \sqrt{61}}{12} \approx 1.56\ldots$ What is known is that $C_d = O(d)$.

In the 1-dimensional case we proved the Hardy-Littlewood inequality using the Rising Sun Lemma, but this has no higher-dimensional analogue. Instead, the main technical tool is:

Finite Vitali Covering Lemma: Let $B_1, \ldots, B_n$ be a finite collection of open balls in $\mathbb{R}^d$. Then $\exists$ a subcollection $B_{i_1}, \ldots, B_{i_k}$ of disjoint balls so that

$$\bigcup_{i=1}^n B_i \subseteq \bigcup_{j=1}^k 3B_{i_j} \implies m\left(\bigcup_{i=1}^n B_i\right) \leq 3^d \sum_{j=1}^k m(B_{i_j})$$

To prove Hardy-Littlewood (with $C_d = 3^d$), it suffices to show $m(K) \leq \frac{3^d}{\lambda} \int_{\mathbb{R}^d} |f(\vec{x})|\, d\vec{x}$ for all compact $K \subseteq \{\vec{x} \in \mathbb{R}^d : \sup_{r>0} \frac{1}{m(B(\vec{x}, r))} \int_{B(\vec{x}, r)} |f(\vec{y})|\, dm(\vec{y}) \geq \lambda\}$.

Now, for each $\vec{x} \in K$, $\exists\, r$ s.t. $\frac{1}{m(B(\vec{x}, r))} \int_{B(\vec{x}, r)} |f(\vec{y})|\, dm(\vec{y}) \geq \lambda$. Take a finite subcover $B_1, \ldots, B_n$, apply Vitali, & conclude that

$$m(K) \leq m\left(\bigcup_{i=1}^n B_i\right) \leq 3^d \sum_{j=1}^k m(B_{i_j}) \leq 3^d \left(\frac{1}{\lambda} \int_{B_{i_j}} |f(\vec{y})|\, dm(\vec{y})\right) \leq \frac{3^d}{\lambda} \int_{\mathbb{R}^d} |f(\vec{x})|\, dm(\vec{x})$$

as desired.

Def: A Borel measure μ on $\mathbb{R}^d$ is **regular** if

① $\mu(K) < \infty$ for all compact K

② $\mu(E) = \inf\{\mu(U) : U \text{ open}, U \supseteq E\}$ for any Borel set E.

Prop: For $f: \mathbb{R}^d \to \mathbb{R}$ w/ $f \geq 0$, the measure m_f is regular $\Leftrightarrow$ $f \in \mathcal{L}^1_{loc}(\mathbb{R}^d, m)$.

Proof: For K cpct, $m_f(K) = \int_K f\,dm \overset{f\geq 0}{=} \int_K |f|\,dm \overset{f \in \mathcal{L}^1_{loc}}{<} \infty$. ☒

Corollary of LDT: Let $\mu = m_f$ for some $f \in \mathcal{L}^1_{loc}$. Then $f(\vec{x}) = \lim_{r\to 0} \frac{\mu(E_r)}{m(E_r)}$ for any family $\{E_r\}_{r>0}$ that shrinks nicely to $\vec{x}$.

In fact, a more general fact is true:

Thm: Let μ be a regular signed or complex Borel measure on $\mathbb{R}^d$, & let $\mu = \lambda + \rho$ w/ $\lambda \perp m$ & $\rho = m_f$ be its Lebesgue-Radon-Nikodym decomposition. Then for a.e. $\vec{x} \in \mathbb{R}^d$,

$$\lim_{r\to 0} \frac{\mu(E_r)}{m(E_r)} = f(\vec{x})$$

for every family $\{E_r\}_{r>0}$ that shrinks nicely to $\vec{x}$.

In this sense, μ is a sort of generalized antiderivative of f.