Math 617: Day 3

Some key features of measures:

Prop: Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:

1. (Monotonicity) If $E, F \in \mathcal{M}$ s.t. $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
2. (Subadditivity) If $E_1, E_2, \ldots \in \mathcal{M}$, then $\mu(E_1 \cup E_2 \cup \ldots) \leq \mu(E_1) + \mu(E_2) + \ldots$
3. (Upwards monotone convergence) If $E_1 \subseteq E_2 \subseteq \ldots$ are measurable, then
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) = \sup_n \mu(E_n)$$
4. (Downward monotone convergence) If $E_1 \supseteq E_2 \supseteq \ldots$ are measurable & $\mu(E_n) < \infty$ for some n, then
$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) = \inf \mu(E_n).$$

Counterexample to ④ if you drop the assumption $\mu(E_n) < \infty$ for some n: Let $X = \mathbb{R}$ & $E_n = [n, \infty)$ where μ is some measure that assigns an interval its length (not that we've shown such a measure exists yet).

Then $\mu(E_n) = \infty$ $\forall n$, but $\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \mu(\emptyset) = 0$.

Proof of Prop: ① Let $E \subseteq F$ be measurable. Note that $F \setminus E = F \cap (E^c)$ is also measurable, &
$$\mu(F) = \mu(E \cup F \setminus E) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

② If the $E_1, E_2, \ldots$ are pairwise disjoint, we're done. Otherwise, define $F_n = E_n \setminus \left(\bigcup_{j=1}^{n-1} E_j\right)$ $\forall n$. Note that, by ①, $\mu(F_n) \leq \mu(E_n)$ & that the F_n are pairwise disjoint. Then
$$\mu(E_1 \cup E_2 \cup \ldots) = \mu(F_1 \cup F_2 \cup \ldots) = \mu(F_1) + \mu(F_2) + \ldots \leq \mu(E_1) + \mu(E_2) + \ldots.$$

③ By ①, $\{\mu(E_n)\}$ is an increasing sequence, so $\lim_{n\to\infty} \mu(E_n) = \sup_n \mu(E_n)$. Now, letting $F_n = E_n \setminus E_{n-1}$,
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{j=1}^{\infty} \mu(F_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(F_j) = \lim_{n\to\infty} \mu(F_1 \cup \ldots \cup F_n) = \lim_{n\to\infty} \mu(E_n).$$

④ By ①, $\{\mu(E_n)\}$ is decreasing, so $\lim_{n\to\infty} \mu(E_n) = \inf_n \mu(E_n)$. Now, by hypothesis, $\mu(E_n) < \infty$ for some n. For $j \geq n$ define $F_j = E_n \setminus E_j$. By ③ we have $\mu\left(\bigcup_{j=n}^{\infty} F_j\right) = \lim_{j\to\infty} \mu(F_j)$. Then
$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \mu\left(\bigcap_{j=n}^{\infty} E_j\right) = \mu\left(E_n \setminus \bigcup_{j=n}^{\infty} F_j\right) = \mu(E_n) - \mu\left(\bigcup_{j=n}^{\infty} F_j\right) = \mu(E_n) - \lim_{j\to\infty} \mu(F_j) = \lim_{j\to\infty} \mu(E_n \setminus F_j) = \lim_{j\to\infty} \mu(E_j)$$

Df: If $(X, \mathcal{M}, \mu)$ is a measure space & $E \in \mathcal{M}$ s.t. $\mu(E) = 0$, then E is a **null set**.

By subadditivity, we have:

Lemma: If $E_1, E_2, \ldots$ are null, then $E_1 \cup E_2 \cup \ldots$ is also null.

If $F \subseteq E$ & E is null, then F is called a **sub-null set**. Unfortunately, sub-null sets aren't necessarily null, since they need not be measurable, but it's usually convenient to modify the σ-algebra of measurable sets to make sub-null sets measurable (& hence null). Such measure spaces are called **complete**.

Prop: Given a measure space $(X, \mathcal{M}, \mu)$, there is a unique completion $(X, \overline{\mathcal{M}}, \bar{\mu})$, where

$$\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M},\ F \text{ sub-null}\}.$$

Proof: If $E \cup F \in \overline{\mathcal{M}}$, then $F \subseteq N$ for some $N \in \mathcal{M}$ s.t. $\mu(N) = 0$. We can assume $E \cap N = \emptyset$; otherwise, use $N \setminus E$ & $F \setminus E$. Then

$$(E \cup F)^c = \left((E \cup N) \cap (N^c \cup F)\right)^c = (E \cup N)^c \cup (N^c \cup F)^c = (E \cup N)^c \cup (N \cap F^c) \in \overline{\mathcal{M}}$$

since $(E \cup N)^c \in \mathcal{M}$ & $N \cap F^c \subseteq N$, which is null.

Also, if $E_1 \cup F_1, E_2 \cup F_2, \ldots \in \overline{\mathcal{M}}$, then

$$(E_1 \cup F_1) \cup (E_2 \cup F_2) \cup \ldots = (E_1 \cup E_2 \cup \ldots) \cup (F_1 \cup F_2 \cup \ldots) \in \overline{\mathcal{M}}$$

since $E_1 \cup E_2 \cup \ldots \in \mathcal{M}$ & $F_1 \cup F_2 \cup \ldots \subseteq N_1 \cup N_2 \cup \ldots$ with $\mu(N_1 \cup N_2 \cup \ldots) \leq \mu(N_1) + \mu(N_2) + \ldots = 0$.

Therefore $\overline{\mathcal{M}}$ is a σ-algebra.

Now, define $\bar{\mu}$ by $\bar{\mu}(E \cup F) = \mu(E)$. This is well-defined: if $E_1 \cup F_1 = E_2 \cup F_2$, then

$$E_1 \cup N_1 \supseteq E_1 \cup F_1 = E_2 \cup F_2 \supseteq E_2, \text{ so } \mu(E_2) \leq \mu(E_1 \cup N_1) = \mu(E_1)$$

Similarly, $\mu(E_1) \leq \mu(E_2)$, so $\mu(E_1) = \mu(E_2)$. But then $\bar{\mu}(E_1 \cup F_1) = \mu(E_1) = \mu(E_2) = \bar{\mu}(E_2 \cup F_2)$.

Moreover, $\bar{\mu}$ is a measure since $\bar{\mu}(\emptyset) = \bar{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0$ &
$$\begin{aligned}\bar{\mu}((E_1 \cup F_1) \cup (E_2 \cup F_2) \cup \ldots) &= \bar{\mu}((E_1 \cup E_2 \cup \ldots) \cup (F_1 \cup F_2 \cup \ldots)) \\ &= \mu(E_1 \cup E_2 \cup \ldots) \\ &= \mu(E_1) + \mu(E_2) + \ldots \\ &= \bar{\mu}(E_1 \cup F_1) + \bar{\mu}(E_2 \cup F_2) + \ldots\end{aligned}$$

Also, $\bar{\mu}$ is complete: if $Y \subseteq E \cup F \in \overline{\mathcal{M}}$ w/ $\bar{\mu}(E \cup F) = \mu(E) = 0$, then $F \subseteq N$ w/ $\mu(N) = 0$, so

$$Y = \emptyset \cup Y \in \overline{\mathcal{M}} \text{ since } Y \subseteq E \cup N.$$

Finally, $\bar{\mu}$ is the unique complete extension of μ to $\overline{\mathcal{M}}$: if η were another, then, $\forall$ $E \cup F \subset \overline{\mathcal{M}}$,

$$\mu(E) = \eta(E) \leq \eta(E \cup F) \leq \eta(E \cup N) = \mu(E \cup N) = \mu(E)$$

so the inequalities have to be equalities & $\eta(E \cup F) = \mu(E) = \bar{\mu}(E \cup F)$. ∎