Math 617: Day 29

All that remains for the Lebesgue differentiation theorem is to prove the Hardy-Littlewood maximal inequality:

Proof of Hardy-Littlewood We're trying to prove $m(\{x\in\mathbb{R} : \sup_{h>0} \frac{1}{h}\int_{[x,x+h]} |f(t)|\,dm(t) \geq \lambda\}) \leq \frac{1}{\lambda}\int_{\mathbb{R}} |f(t)|\,dm(t)$.

Of course, it's enough to show $m(\{x\in[a,b] : \sup_{h>0,\,[x,x+h]\subseteq[a,b]} \frac{1}{h}\int_{[x,x+h]} |f(t)|\,dm(t) \geq \lambda\}) \leq \frac{1}{\lambda}\int_{\mathbb{R}} |f(t)|\,dm(t)$ for any $[a,b]$.

Now, the idea is to cook up a function which is in shadow when $\frac{1}{h}\int_{[x,x+h]} |f(t)|\,dm(t) \geq \lambda$ & then use countable additivity to bound the size of the set where this happens.

Define $F:[a,b]\to\mathbb{R}$ by $F(x) := \int_{[a,x]} |f(t)|\,dm(t) - (x-a)\lambda$. Notice that

$$F(x+h)-F(x) = \left(\int_{[a,x+h]} |f(t)|\,dm(t) - (x+h-a)\lambda\right) - \left(\int_{[a,x]} |f(t)|\,dm(t) - (x-a)\lambda\right) = \int_{[x,x+h]} |f(t)|\,dm(t) - h\lambda, \text{ so}$$

$$\frac{1}{h}\int_{[x,x+h]} |f(t)|\,dm(t) \leq \lambda \Rightarrow F(x+h) \geq F(x).$$

So now, apply the rising sun lemma to F to get the disjoint intervals $I_n = (a_n, b_n)$. Then, by the above observation,

$$\{x\in[a,b) : \sup_{h>0,\,[x,x+h]\subseteq[a,b)} \frac{1}{h}\int_{[x,x+h]} |f(t)|\,dm(t) \geq \lambda\} \subseteq \bigsqcup I_n,$$

so the measure of the LHS is $\leq m(\bigsqcup I_n) = \sum_n m(I_n) = \sum_n (b_n - a_n)$.

OTOH,

$$0 \leq F(b_n) - F(a_n) = \ldots = \int_{I_n} |f(t)|\,dm(t) - (b_n - a_n)\lambda,$$

so $b_n - a_n \leq \frac{1}{\lambda}\int_{I_n} |f(t)|\,dm(t)$.

But then

$$\{x\in[a,b) : \sup_{h>0,\,[x,x+h]\subseteq[a,b)} \frac{1}{h}\int_{[x,x+h]} |f(t)|\,dm(t) \geq \lambda\} \leq \sum (b_n - a_n) \leq \sum \frac{1}{\lambda}\int_{I_n} |f(t)|\,dm(t) \leq \frac{1}{\lambda}\int_{\mathbb{R}} |f(t)|\,dm(t),$$

as desired. ∎

Higher Dimensions

There's not really a direct generalization of the Lebesgue differentiation theorem as stated to higher dimensions, but there is a generalization of our equations ⊛ & ⊛⊛, which really said

$$\lim_{h\to 0^+} \frac{1}{h}\int_{[x,x+h]} f(t)\,dm(t) = f(x) \quad \text{for a.e. } x\in\mathbb{R}$$

$$\lim_{h\to 0^+} \frac{1}{h}\int_{[x-h,x]} f(t)\,dm(t) = f(x) \quad \text{for a.e. } x\in\mathbb{R}.$$

Namely,

Higher-Dimensional Lebesgue Differentiation Thm: Let $f:\mathbb{R}^d \to \mathbb{C}$ be absolutely integrable. Then for a.e. $\vec{x} \in \mathbb{R}^d$,

$$\lim_{r\to 0} \frac{1}{m(B(\vec{x},r))} \int_{B(\vec{x},r)} |f(y)-f(\vec{x})|\, dm(y) = 0 \quad \& \quad \lim_{r\to 0} \frac{1}{m(B(\vec{x},r))} \int_{B(\vec{x},r)} f(y)\, dm(y) = f(x).$$

Now,

$$\left| \frac{1}{m(B(\vec{x},r))} \int_{B(\vec{x},r)} f(y)\, dm(y) - f(x) \right| = \left| \frac{1}{m(B(\vec{x},r))} \int_{B(\vec{x},r)} (f(y)-f(x))\, dm(y) \right| \le \frac{1}{m(B(\vec{x},r))} \int_{B(\vec{x},r)} |f(y)-f(x)|\, dm(y),$$

So the first conclusion in the theorem implies the second.

In fact, the theorem is still true if you replace "absolutely integrable" w/ "locally integrable":

Df: A function $f:\mathbb{R}^d \to \mathbb{C}$ is **locally integrable** if $\forall\, x \in \mathbb{R}$ $\exists$ open nbhd U of x s.t. f is absolutely integrable on U.

The set of locally integrable functions is denoted $L^1_{loc}(\mathbb{R}^d, m)$ or sometimes just L^1_{loc}.

Prop: $f \in L^1_{loc} \iff \int_{B(\vec{0},r)} |f(\vec{x})|\, dm(\vec{x}) < \infty \;\; \forall\, r>0 \iff \int_K |f(\vec{x})|\, dm(\vec{x}) < \infty \;\; \forall$ bdd, measurable sets K.