Math 617: Day 28

We're in the midst of proving the 1-d Lebesgue differentiation theorem, which recall was equivalent to showing

$$(*) \qquad \lim_{h\to 0^+} \frac{1}{h}\int_{[x,x+h]} f(t)\,dm(t) = f(x) \quad \text{for almost any } x\in\mathbb{R}$$

Now, we've seen this is true for continuous, compactly supported functions, & we've seen that for any absolutely integrable f, $\exists$ continuous, compactly supported g s.t. $\|f-g\|_{L^1} \le \varepsilon$.

On the other hand, the quantitative estimate we need is:

Lemma (One-sided Hardy-Littlewood maximal inequality): Let $f:\mathbb{R}\to\mathbb{C}$ be absolutely integrable & let $\lambda>0$. Then

$$m\left(\left\{x\in\mathbb{R} : \sup_{h>0} \frac{1}{h}\int_{[x,x+h]} |f(t)|\,dm(t) \ge \lambda\right\}\right) \le \frac{1}{\lambda}\int_{\mathbb{R}} |f(t)|\,dm(t)$$

We'll prove this shortly, but first let's see why the Lebesgue differentiation theorem follows:

Proof of Lebesgue Differentiation Thm: Let $f:\mathbb{R}\to\mathbb{C}$ be absolutely integrable & let $\varepsilon,\lambda>0$.

By Littlewood's second principle, $\exists$ continuous, compactly supported $g:\mathbb{R}\to\mathbb{C}$ s.t.

$$\int_{\mathbb{R}} |f-g|\,dm \le \varepsilon$$

Using the Hardy-Littlewood inequality,

$$m\left(\left\{x\in\mathbb{R} : \sup_{h>0} \frac{1}{h}\int_{[x,x+h]} |f(t)-g(t)|\,dm(t) \ge \lambda\right\}\right) \le \frac{1}{\lambda}\int_{\mathbb{R}} |f-g|\,dm \le \frac{\varepsilon}{\lambda}.$$

OTOH,

$$m\left(\left\{x\in\mathbb{R} : |f(x)-g(x)|\ge\lambda\right\}\right) \le \frac{1}{\lambda}\int |f(x)-g(x)|\,dm(x) \le \frac{\varepsilon}{\lambda} \qquad \text{(why?)}$$

Let E be the union of the these two sets; we know $m(E)\le \frac{2\varepsilon}{\lambda}$. On E^c we have both

$$\frac{1}{h}\int_{[x,x+h]} |f(t)-g(t)|\,dm(t) < \lambda \quad \& \quad |f(x)-g(x)|<\lambda \qquad \forall\, h>0.$$

Now, if $x\in E^c$, we know $\left|\frac{1}{h}\int_{[x,x+h]} g(t)\,dm(t) - g(x)\right| < \lambda$ for h sufficiently small, since g is continuous & compactly supported.

So then

$$\begin{aligned}
\left|\frac{1}{h}\int_{[x,x+h]} f(t)\,dm(t) - f(x)\right| &= \left|\frac{1}{h}\int_{[x,x+h]} f(t)\,dm(t) - \frac{1}{h}\int_{[x,x+h]} g(t)\,dm(t) + \frac{1}{h}\int_{[x,x+h]} g(t)\,dm(t) - g(x) + g(x) - f(x)\right| \\
&\le \left|\frac{1}{h}\int_{[x,x+h]} (f(t)-g(t))\,dm(t)\right| + \left|\frac{1}{h}\int_{[x,x+h]} g(t)\,dm(t) - g(x)\right| + |g(x)-f(x)| \\
&< \frac{1}{h}\int |f(t)-g(t)|\,dm(t) + \lambda + \lambda \\
&< \lambda+\lambda+\lambda = 3\lambda \quad \text{for sufficiently small } h.
\end{aligned}$$

This implies $\limsup_{h\to 0} \left|\frac{1}{h}\int_{[x,x+h)} f(t)\,dm(t) - f(x)\right| < 3\lambda \quad \forall x\in E^c$. Now, as $\varepsilon\to 0$, since $m(E)\le \frac{2\varepsilon}{\lambda}$, we see

$$\limsup_{h\to 0} \left|\frac{1}{h}\int_{[x,x+h)} f(t)\,dm(t) - f(x)\right| < 3\lambda \quad \text{for almost any } x\in\mathbb{R}.$$

Now, letting $\lambda\to 0$ gives that

$$\limsup_{h\to 0} \left|\frac{1}{h}\int_{[x,x+h)} f(t)\,dm(t) - f(x)\right| = 0 \quad \text{for almost any } x\in\mathbb{R}.$$

This completes the proof of Ⓐ &, hence, of the Lebesgue differentiation theorem. $\boxtimes$

The proof of the Hardy-Littlewood maximal inequality will use the:

Rising Sun Lemma: Let $[a,b]$ be a compact interval & let $F:[a,b]\to\mathbb{R}$ be continuous. Then $\exists$ a finite or countable family of disjoint, nonempty open intervals $I_n=(a_n,b_n)$ s.t.

① $\forall n$, either $F(a_n)=F(b_n)$ or $a_n=a$ & $F(b_n)\ge F(a_n)$.

② If $x\in(a,b]$ does not lie in any of the I_n, then $F(x)\ge F(y)$ $\forall$ $x\le y\le b$.

The idea: Draw the graph, & have the sun shining from $+\infty$. Then the intervals are the points in shade: