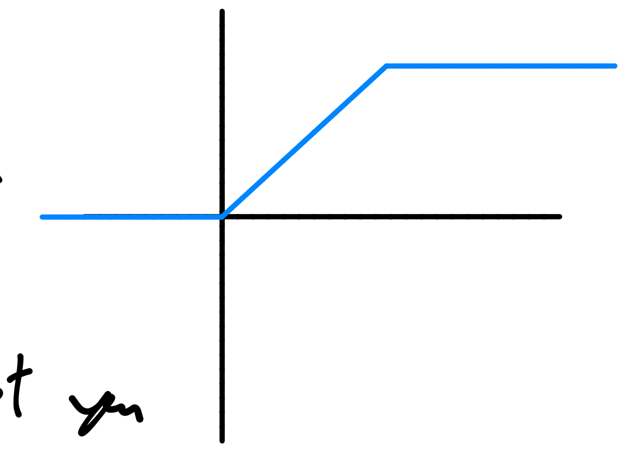
# Math 417: Day 27

The goal for the next couple of days is to prove:

**1-Dimensional Lebesgue Differentiation Theorem:** Let $f:\mathbb{R}\to\mathbb{C}$ be absolutely integrable (w.r.t. Lebesgue measure) & let $F:\mathbb{R}\to\mathbb{C}$ be defined by $F(x) := \int_{(-\infty,x]} f(t)\,dm(t)$. Then $F$ is continuous & differentiable almost everywhere, with $F'(x)=f(x)$ a.e.

**Ex:** Consider $f=\chi_{[0,1]}$. Then $F$ is the function



This makes it clear that diff'ble a.e. is the best you can do.

Note that you already proved continuity in HW3 #1.

Now, the actual strategy is to prove the following equivalent fact:

**Prop 1:** Let $f:\mathbb{R}\to\mathbb{C}$ be absolutely integrable. Then

$$(\ast)\qquad \lim_{h\to 0^+} \frac{1}{h}\int_{[x,x+h]} f(t)\,dm(t) = f(x)$$

for almost any $x\in\mathbb{R}$ &

$$(\ast\ast)\qquad \lim_{h\to 0^+} \frac{1}{h}\int_{[x-h,x]} f(t)\,dm(t) = f(x)$$

for almost any $x\in\mathbb{R}$.

Now, it's enough to just prove $(\ast)$, since $(\ast\ast)$ follows by using $g(x)=f(-x)$ in $(\ast)$.

Also, by applying to real & imaginary parts, we may as well assume $f$ is real-valued.

The approach is a kind of density argument: prove for a dense subclass, then get quantitative bounds on the size of $\frac{1}{h}\int_{[x,x+h]} f(t)\,dm(t)$ in terms of $\int_{\mathbb{R}} |f|\,dm$ & do some triangle inequality magic.

In this setting, the dense subclass will be the continuous functions:

**Prop:** If $f:[a,b]\to\mathbb{C}$ is continuous, then

- $\lim_{h\to 0^+} \frac{1}{h}\int_{[x,x+h]} f(t)\,dm(t) = f(x) \quad \forall\, x\in[a,b)$

- $\lim_{h\to 0^+} \frac{1}{h}\int_{[x-h,x]} f(t)\,dm(t) = f(x) \quad \forall\, x\in(a,b]$

and hence
- $\lim_{h\to 0^+} \frac{1}{2h}\int_{[x-h,x+h]} f(t)\,dm(t) = f(x) \quad \forall\, x\in(a,b)$.

Of course, this is just a version of the usual Fundamental Thm of Calculus (since the Lebesgue integral & the Riemann integral are the same for continuous functions)

In what sense are continuous, compactly supported functions dense among the absolutely integrable functions?

Well, that's the content of Littlewood's Second Principle. Here are Littlewood's three principles:

① Every measurable set is almost a finite union of intervals

② Every absolutely integrable function is nearly continuous

③ Every pointwise convergent sequence of functions is nearly uniformly convergent

① is basically the fact that inner & outer measure agree for measurable sets. ③ is just Egorov's Thm. More precisely, ② says

**Prop:** Let $f\in L^1(\mathbb{R},m)$ & $\varepsilon>0$. Then

① $\exists$ absolutely integrable simple function $g$ s.t. $\|f-g\|_{L^1} = \int_{\mathbb{R}} |f(x)-g(x)|\,dm(x) \le \varepsilon$

② $\exists$ step function $g$ s.t. $\|f-g\|_{L^1} \le \varepsilon$

③ $\exists$ continuous, compactly supported $g$ s.t. $\|f-g\|_{L^1}\le\varepsilon$

**Proof:** ① This is obviously true for $f\ge 0$, since such $f$ are an increasing limit of simple functions. Of course, by taking positive & negative parts & using the triangle inequality, it's true for real-valued $f$, & in turn by taking real & imaginary parts & using the triangle inequality, it's true for complex-valued $f$.

② If ② is true for characteristic functions $\chi_E$, then we see that $\exists$ simple $g=\sum_{i=1}^n c_i\chi_{E_i}$ s.t. $\|f-g\|_{L^1}\le \varepsilon/2$ & $\exists$ step functions $g_i$ s.t. $\|\chi_{E_i}-g_i\|\le \frac{\varepsilon}{2|c_i|n}$, so $\|f-\sum c_i g_i\|_{L^1} = \|f-g+g-\sum c_i g_i\| \le \|f-g\| + \sum |c_i|\,\|\chi_{E_i}-g_i\|$

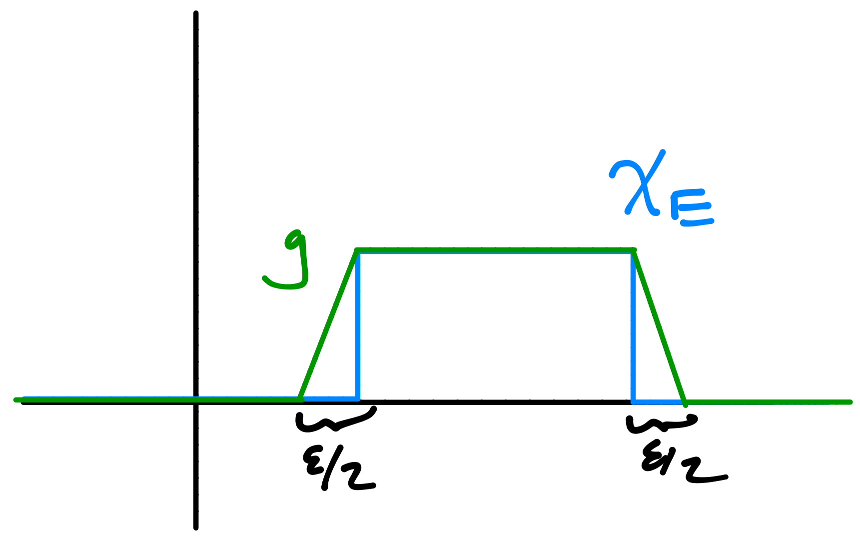
$$\le \varepsilon/2 + \sum \frac{|c_i|}{2|c_i|n}$$

$$= \varepsilon.$$

But now, if $f=\chi_E$, then, since $m(E)=m^*(E)$, $\exists$ an elementary set $U\supseteq E$ s.t. $m(U\setminus E)\le\varepsilon$. But then

$\chi_U$ is a step function & $\|\chi_U-\chi_E\| = \int_{\mathbb{R}}|\chi_U-\chi_E|\,dm(t) = \int_{U\setminus E} dm = m(U\setminus E)\le\varepsilon$.

③ Again, by the same reasoning, it's enough to prove this for $f = \chi_E$, where $E$ is an interval. But now just let $g$ be the function whose graph is




Then $g$ is continuous & $\|f-g\| \le \varepsilon$.