Math 417: Day 26

So far, we've seen that $\nu \ll \mu \iff \nu = \mu_f$ for some $f = \frac{d\nu}{d\mu}$. This is sometimes also denoted

$$d\nu = \frac{d\nu}{d\mu}\, d\mu$$

since $\nu(E) = \int_E d\nu = \int_E \frac{d\nu}{d\mu}\, d\mu$.

This derivative notation isn't entirely a lie. For example, we have some sort of chain rule:

Prop: If μ, λ are σ-finite positive measures & ν is a σ-finite signed measure on $(X, \mathcal{M})$ with $\nu \ll \mu$ & $\mu \ll \lambda$, then

$$\nu \ll \lambda \quad \& \quad \frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Proof: For any $E \in \mathcal{M}$, $\nu(E) = \int_E \frac{d\nu}{d\mu}\, d\mu$. Now, we know that for any measurable g, $\int_E g\, d\mu = \int_E g \frac{d\mu}{d\lambda}\, d\lambda$ so, letting $g = \frac{d\nu}{d\mu}$,

$$\nu(E) = \int_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}\, d\lambda.$$

But then this means $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$ λ-a.e.

Cor: $\frac{d\nu}{d\mu}\frac{d\mu}{d\nu} = 1$ a.e.

Complex Measures

Essentially all theorems we know also hold for measures that are allowed to assign complex values to sets.

Def: A **complex measure** on $(X, \mathcal{M})$ is a map $\mu : \mathcal{M} \to \mathbb{C}$ s.t.

1. $\mu(\emptyset) = 0$
2. $\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$ where the sequence converges absolutely.

Notice, in particular, that complex measures are not allowed to assign infinite measure to sets.

A complex measure μ has a **real part** μ_r & an imaginary part μ_i & we define $L^1(X, \mathcal{M}, \mu) = L^1(X, \mathcal{M}, \mu_r) \cap L^1(X, \mathcal{M}, \mu_i)$ with

$$\int f\, d\mu = \int f\, d\mu_r + i \int f\, d\mu_i.$$

If μ is a positive measure & ν is a complex measure, then $\nu \ll \mu$ if $\nu_r \ll \mu$ & $\nu_i \ll \mu$.

Now, the Lebesgue-Radon-Nikodym theorem is true for complex measures (by applying the signed LRNT to the real & imaginary parts)