Math 617: Day 25

Now, we saw before that for any measurable $f: X \to [0,+\infty]$ on $(X, \mathcal{M}, \mu)$, we get a positive measure μ_f given by

$$\mu_f(E) = \int_E f \, d\mu.$$

Similarly, if $f \in L^1(X, \mathcal{M}, \mu)$, then we get a finite signed measure $\mu_f = \mu_{f_+} - \mu_{f_-}$. This defines an embedding $L^1(X,\mathcal{M},\mu) \to \mathcal{M}(X)$.

Now, it's not true that any $\nu \in \mathcal{M}(X)$ is equal to μ_f for some $f \in L^1(X)$. For example, if $\nu = \mu_f$, then for any $E \in \mathcal{M}$ w/

$\mu(E) = 0$, it would have to be the case that $\nu(E) = \int_E f \, d\mu = 0$ as well.

So, for example, if $\nu \perp \mu$, there's no way that $\nu = \mu_f$. In fact, this is the only obstruction:

Lebesgue-Radon-Nikodym Thm: Let μ be a σ-finite positive measure on $(X, \mathcal{M})$ & let ν be a signed σ-finite measure. Then there exists a unique decomposition

$$\nu = \lambda + \rho$$

where $\lambda \perp \mu$ & $\rho = \mu_f$ for $f: X \to \mathbb{R}$ measurable. If ν is finite, so are f & λ, and if ν is finite, $f \in L^1(X)$ & λ finite.

Proof: We prove this just for μ & ν finite. For uniqueness, suppose $\nu = \lambda' + \rho'$ w/ $\rho' = \mu_g$. But now, by the above observation,

$\mu(E) = 0 \Rightarrow \rho(E) = 0$ & $\rho'(E) = 0$, so $\rho(E) = \rho'(E)$ $\forall E$. Similarly, $\lambda(E) = \lambda'(E)$, so we see λ & ρ are unique (if they exist).

Moreover, it's easy to see that $\mu_f = \mu_g \Leftrightarrow f = g$ μ-a.e., so f is unique up to μ-a.e. equivalence.

Since we can write $\nu = \nu_+ - \nu_-$ & prove for ν_+ & ν_- separately, we may as well assume ν is positive.

Of course, the challenge is to find the function f. Let $M = \sup\{\int_X f \, d\mu : f \geq 0, \mu_f \leq \nu\}$. Since μ is finite, so is M.

In fact, I claim M is a max: if $(f_n)_{n=1}^{\infty}$ is a maximizing sequence, then $\mu_{f_n} \leq \nu$ & $\int_X f_n \, d\mu \to M$, so define $f := \sup_n f_n$. Then $\int_X f \, d\mu = M$ & $\mu_f \leq \nu$.

Now, let $\rho = \mu_f$ & define $\lambda := \nu - \rho$. Then λ is positive & finite, so we need to show $\lambda \perp \mu$.

If not, then by the following lemma $\exists$ $E \in \mathcal{M}$ & $\varepsilon > 0$ s.t. $\mu(E) > 0$ & $\lambda \geq \varepsilon\mu$ on E.

But then $\mu_{\varepsilon\chi_E} \leq \lambda = \nu - \mu_f$, so $\mu_{f+\varepsilon\chi_E} = \mu_f + \mu_{\varepsilon\chi_E} \leq \nu$, so $f + \varepsilon\chi_E$ is included in the sup defining M.

This is a problem: $\int_X (f + \varepsilon\chi_E) \, d\mu = \int_X f \, d\mu + \int_X \varepsilon\chi_E \, d\mu = M + \varepsilon\mu(E) > M$, contradicting the fact that M is the sup.

From the contradiction, we see that, in fact, $\lambda \perp \mu$, as desired. ∎

Lemma: Suppose α & β are finite positive measures on $(X, \mathcal{M})$. Then either $\alpha \perp \beta$ or $\exists$ $\varepsilon > 0$ & $E \in \mathcal{M}$ s.t. $\beta(E) > 0$

& $\alpha \geq \varepsilon \beta$ on E.

Proof: See Lemma 3.7 in Folland. ☒

Corollary: Let μ be a positive, σ-finite measure on $(X, \mathcal{M})$ & let ν be a signed, σ-finite measure. TFAE:

① $\nu = \mu_f$ for some measurable f.

② $\nu(E) = 0$ whenever $\mu(E) = 0$.

③ $\forall$ $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $\mu(E) \leq \delta \Rightarrow |\nu(E)| < \varepsilon$.

Def: When any of the statements ①, ②, or ③ from the Corollary are true, we say that ν is *absolutely continuous* with respect to μ, denoted

$\nu \ll \mu$, and the function $f =: \frac{d\nu}{d\mu}$ is the *Radon-Nikodym derivative* of ν w.r.t. μ.