# Math 617: Day 24

Now, the Hahn decomposition is not unique, since we can freely pass null sets back & forth between $X_+$ & $X_-$, but it does lead to a unique decomposition of signed measures into positive & negative parts.

**Def**: A measure $\mu$ on $(X,\mathcal{M})$ is **supported** on $E\in\mathcal{M}$ if $\mu(E^c)=\emptyset$.

Two signed measures $\mu$ & $\nu$ on $(X,\mathcal{M})$ are **mutually singular**, denoted $\mu\perp\nu$, if they are supported on disjoint sets.

Equivalently, $\mu\perp\nu \iff \exists\, E,F\in\mathcal{M}$ s.t. $E\cap F=\emptyset$, $E\cup F=X$, & $\mu(F)=\emptyset=\nu(E)$.

Then we get a decomposition of signed measures into positive & negative parts, just like we did for functions:

**Jordan Decomposition Thm**: If $\mu$ is a signed measure on $(X,\mathcal{M})$, then there exist unique **positive** measures $\mu_+$ & $\mu_-$ with

$\mu_+\perp\mu_-$ & $\mu=\mu_+-\mu_-$.

**Proof**: Let $X_+\cup X_-$ be a Hahn decomposition of $X$ w.r.t. $\mu$. For $E\in\mathcal{M}$, define $\mu_+(E):=\mu(E\cap X_+)$ & $\mu_-(E)=-\mu(E\cap X_-)$.

Then clearly $\mu_+$ & $\mu_-$ are both positive measures, $\mu_+(E)-\mu_-(E)=\mu(E\cap X_+)+\mu(E\cap X_-)=\mu(E)$ for any $E\in\mathcal{M}$,

and $\mu_+\perp\mu_-$.

Now, suppose $\mu=\nu_+-\nu_-$ where $E,F\in\mathcal{M}$, $E\cap F=\emptyset$, $E\cup F=X$, $\nu_+(F)=\emptyset=\nu_-(E)$. Then $E\cup F$ is an alternative

Hahn decomposition of $X$. But $X_+\setminus E\subseteq X_+$ is positive & $X_+\setminus E\subseteq F$ is negative, so $\mu(X_+\setminus E)=0$.

Similarly, $\mu(E\setminus X_+)=0$. Therefore, for any $A\in\mathcal{M}$, $\nu_+(A)=\nu_+(A\cap E)=\mu(A\cap E)=\mu(A\cap X_+)=\mu_+(A)$

because $A\cap E=(A\cap X_+\cap E)\cup(A\cap(E\setminus X_+))=((A\cap X_+)\setminus(A\cap(X_+\setminus E)))\cup(A\cap(E\setminus X_+))$

so $\mu(A\cap E)=\mu(A\cap X_+)-\mu(A\cap(X_+\setminus E))+\mu(A\cap(E\setminus X_+))=\mu(A\cap X_+)-0+0$

since $A\cap(X_+\setminus E)\subseteq X_+\setminus E$ & $A\cap(E\setminus X_+)\subseteq E\setminus X_+$ are both null.

Likewise, $\nu_-(A)=\mu_-(A)$ $\forall A\in\mathcal{M}$, so we see that the decomposition is unique. ▨

Now, just as the absolute value of a function $f=f_+-f_-$ can be given as $|f|=f_++f_-$, we can define the **absolute value** or **total variation** of a signed measure $\mu=\mu_+-\mu_-$ to be $|\mu|=\mu_++\mu_-$.

**Def**: A signed measure $\mu$ is **finite** (respectively, **$\sigma$-finite**) if $|\mu|$ is a finite (respectively, $\sigma$-finite) positive measure.

The set of all finite signed measures on $X$ is a vector space, denoted $\mathcal{M}(X)$.

Prop: Let $\mu$ be a signed measure on $(X, \mathcal{M})$. TFAE:

① $\mu$ is finite.

② $\mu(E)$ is finite $\forall$ $E \in \mathcal{M}$.

③ $\mu_+$ & $\mu_-$ are both finite positive measures.