Math 617: Day 23

Of course, if $f: X \to [-\infty, +\infty]$ is a function, we can write $X = X_+ \cup X_-$ where $f \geq 0$ on X_+, $f < 0$ on X_-, & $X_+ \cap X_- = \emptyset$.

The same applies to signed measures:

Hahn Decomposition Thm: Let μ be a signed measure on $(X, \mathcal{M})$. Then $\exists\, X_+, X_- \in \mathcal{M}$ s.t.

① $X_+ \cup X_- = X$ ② $X_+ \cap X_- = \emptyset$ ③ $\mu|_{X_+} \geq 0$ & $\mu|_{X_-} \leq 0$.

Proof: First off, we can assume μ avoids $+\infty$ by replacing it with $-\mu$, if necessary.

Now, we say $E \in \mathcal{M}$ is *positive* if $\mu|_E \geq 0$, & E is *negative* if $\mu|_E \leq 0$.

The goal is to build X_+ to be the biggest possible positive set:

Let $M_+ = \sup\{\mu(E) : E \text{ positive}\}$. M_+ exists since $\emptyset \in \{\mu(E) : E \text{ positive}\}$ & is finite since $\mu < +\infty$.

In fact, I claim M_+ is attained. To see this, note that $\exists\, E_1, E_2, \ldots \in \mathcal{M}$ positive s.t.

$$M_+ = \lim_{n \to \infty} \mu(E_n).$$

But now let $X_+ = \bigcup_{n=1}^{\infty} E_n$. Then X_+ is positive by construction &

$$\mu(X_+) = \mu\Big(\bigcup_{n=1}^{\infty} \tilde{E}_n\Big) = \sum_{n=1}^{\infty} \mu(\tilde{E}_n) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(E_i) = M_+$$

where $\tilde{E}_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$.

Now, let $X_- = X \setminus X_+$. Clearly X_+ & X_- satisfy ① & ②, so all that's left is to show X_- is negative.

If not, then $\exists\, P_1 \subseteq X_-$ with $\mu(P_1) > 0$. If P_1 is positive, then $X_+ \cup P_1$ is positive &

$$\mu(X_+ \cup P_1) = \mu(X_+) + \mu(P_1) = M_+ + \mu(P_1) > M_+,$$ contradicting the fact that M_+ is maximal.

OTOH, if P_1 is *not* positive, then $\exists\, N_1 \subseteq P_1$ s.t. $\mu(N_1) < 0$. Let n_1 be the smallest integer for which there exists such an N_1 w/ $\mu(N_1) < \frac{-1}{n_1}$. Then, if $P_2 = P_1 \setminus N_1$, we have $\mu(P_2) \geq \mu(P_1) + \frac{1}{n_1}$.

But now P_2 is either positive or not. If it is, we again have a contradiction. Otherwise, P_2 contains P_3 w/ $\mu(P_3) > \mu(P_2) + \frac{1}{n_2}$, where n_2 is the smallest positive integer for which this is possible.

Iterating, either we halt at some stage or we get a decreasing sequence of sets $P_1 \supseteq P_2 \supseteq \ldots$ w/

$$\mu(P_{j+1}) \geq \mu(P_j) + \frac{1}{n_j}$$

Now, if $P = \bigcap_{n=1}^{\infty} P_n$, then $\infty > \mu(P) = \lim_{n\to\infty} \mu(P_n) > \sum_{k=1}^{\infty} \frac{1}{n_k}$, so the sequence $n_1, n_2, \ldots$ must go to infinity.

But this means P is positive. If it weren't, it would contain $\tilde{P}$ w/ $\mu(\tilde{P}) \geq \mu(P) + \frac{1}{n}$ for some n.

But then $\exists k$ w/ $n_k > n$, so $\tilde{P} \subseteq P_k$ & $\mu(\tilde{P}) \geq \mu(P) + \frac{1}{n} \geq \mu(P_k) + \frac{1}{n}$, meaning n_k wasn't actually the smallest such integer & we finally have our contradiction & conclude X_- is negative. ∎