Math 617: Day 22

We are now going to talk about differentiation in the context of measures. One of the ultimate goals is to prove a version of the Fundamental Theorem of Calculus for Lebesgue measure on $\mathbb{R}$ & even $\mathbb{R}^n$.

It turns out that the "antiderivative" of a (locally) integrable function f will actually be a measure.

In some sense we've already seen this: for a measure space $(X, \mathcal{M}, \mu)$ & a measurable function $f: X \to [0, +\infty]$, we get a new measure μ_f defined by

$$\mu_f(E) := \int_E f \, d\mu.$$

And we just saw that for any measurable g, $\int_X g \, d\mu_f = \int_X g f \, d\mu$, so we symbolically write $d\mu_f = f d\mu$ or, playing fast & loose w/ notation,

$$\frac{d\mu_f}{d\mu} = f.$$

In general, we will say that a measure ν is differentiable w.r.t. μ if $\exists \, f: X \to [0, +\infty]$ s.t. $d\nu = f d\mu$, & we call

$$f =: \frac{d\nu}{d\mu}$$

the Radon-Nikodym derivative of ν w.r.t. μ.

And so, again playing fast & loose w/ notation, $\nu = \int f \, d\mu$, where this is the "indefinite" integral.

Prop: Let m be Lebesgue measure on $[0, +\infty)$ & let μ be a measure which is differentiable w.r.t. m. If $\frac{d\mu}{dm}$ is continuous, then $F_\mu : [0, +\infty) \to [0, +\infty]$ given by $F_\mu(x) := \mu([0, x])$ is differentiable, & $\frac{d}{dx} F_\mu(x) = \frac{d}{dx} \mu([0, x]) = \frac{d\mu}{dm}(x) \; \forall x$.

Now, the problem is that not every measure is differentiable w.r.t. a given measure (e.g., Dirac measure is not diff'ble w.r.t. Lebesgue measure), but it will turn out that, for any measures μ & ν,

$$\nu = \lambda + \rho$$

where λ is supported (kind of) on the complement of the support of μ & ρ is diff'ble w.r.t. μ, which is really as good as we could have expected.

This fact is called the Lebesgue-Radon-Nikodym theorem, but to prove it we need to be able to do things like write

$\lambda = \nu - \rho$.

Of course adding measures is no problem: for μ & ν measures, we can define $\mu+\nu$ by $(\mu+\nu)(E) = \mu(E)+\nu(E)$,

and indeed, $\mu_{f+g} = \mu_f + \mu_g$.

But an expression like $\nu-\rho$ may very well not always be positive, so first we need to introduce the concept of

signed measures.

The idea is that a signed measure on a measurable space $(X,\mathcal{M})$ should be a map $\mu: \mathcal{M} \to [-\infty,\infty]$.

What are the possible issues w/ an expression like $\nu-\rho$, which should be defined by $(\nu-\rho)(E) = \nu(E)-\rho(E)$?

① If $\nu(E)=\infty$ & $\nu(E)=\infty$, $\infty-\infty$ is undefined. We will fix this by only allowing $\nu-\rho$ if at least one is a finite measure.

This will mean $\nu-\rho$ takes values in either $[-\infty,+\infty)$ or $(-\infty,+\infty]$, but not both.

② $\nu-\rho$ is likely not to be monotone.

These considerations lead to the formal definition:

Df: A signed measure on $(X,\mathcal{M})$ is a map $\mu: \mathcal{M} \to [-\infty,+\infty]$ s.t.

① $\mu(\emptyset)=0$

② μ can take the value $+\infty$ or $-\infty$, but not both.

③ If $E_1, E_2, \ldots \in \mathcal{M}$ are disjoint, then $\sum_{i=1}^{\infty} \mu(E_i)$ converges to $\mu(\bigcup_{i=1}^{\infty} E_i)$, & the convergence is absolute if the RHS is finite

Ex: If $(X,\mathcal{M},\mu)$ is a measure space & $f: X \to [-\infty,+\infty]$ is measurable, then μ_f is a signed measure if at least one of f_+ or f_-

is absolutely integrable.

We sometimes call measures positive measures when we want to emphasize that they're special cases of signed measures.