Math 617: Day 2

As discussed last time, given a set X on which we want to define a measure, we need to specify which subsets of X are going to be **measurable**. The collection of measurable sets **cannot** be arbitrary... it turns out to have **algebraic** structure.

We can guess what this structure must be. Surely the whole set X should be measurable, and, if $E \subseteq X$ is measurable, then

$E^c = X \setminus E$ should also be measurable (since $\mu(E^c) = \mu(X \setminus E) = \mu(X) - \mu(E)$).

Also, if $E_1, E_2, \ldots \subseteq X$, then $E_1 \cup E_2 \cup \ldots$ should also be measurable (since $\mu(E_1 \cup E_2 \cup \ldots) = \mu(E_1) + \mu(E_2) + \ldots$)

Df: If X is a nonempty set, then a **σ-algebra** on X is a nonempty collection $\mathcal{A}$ of subsets of X so that:

① For $E_1, E_2, \ldots \in \mathcal{A}$, $E_1 \cup E_2 \cup \ldots \in \mathcal{A}$ (closed under countable unions)

② For $E \in \mathcal{A}$, $E^c \in \mathcal{A}$ (closed under complementation)

Prop: A σ-algebra $\mathcal{A}$ on X

① is closed under countable intersections

② contains $\emptyset$ & X

Pf: ① For $E_1, E_2, \ldots \in \mathcal{A}$, $E_1 \cap E_2 \cap \ldots = (E_1 \cup E_2 \cup \ldots)^c \in \mathcal{A}$

② For $E \in \mathcal{A}$ (which exists since $\mathcal{A}$ is nonempty), $\emptyset = E \cap E^c \in \mathcal{A}$ & $X = E \cup E^c \in \mathcal{A}$. ▨

Ex: ① $\{\emptyset, X\}$ is a σ-algebra (the **trivial** σ-algebra)

② $\mathcal{P}(X)$ is a σ-algebra (the **discrete** σ-algebra)

Prop: Let I be some index set (possibly uncountable) &, for each $\alpha \in I$, suppose $\mathcal{A}_\alpha$ is a σ-algebra on a set X. Then

$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha \text{ is a } \sigma\text{-algebra on } X.$$

Pf: HW. ▨

Therefore, given any collection $\mathcal{E}$ of subsets of X, there is a unique smallest σ-algebra $\mathcal{M}(\mathcal{E})$ containing $\mathcal{E}$: $\mathcal{M}(\mathcal{E})$ is the intersection of **all** σ-algebras containing $\mathcal{E}$ (which exists, since $\mathcal{E} \subseteq \mathcal{P}(X)$, which is a σ-algebra). $\mathcal{M}(\mathcal{E})$ is called the σ-alg. **generated by** $\mathcal{E}$.

Remark: A σ-algebra is a special case of a **Boolean algebra**, which is a set $\mathcal{B}$ together w/ operations $\wedge$ & $\vee$ satisfying the axioms you'd expect from symbolic logic. In turn, Boolean algebras correspond to **Boolean rings**, which are rings w/ $x \cdot x = x \ \forall x$.

In other words, we can turn a σ-algebra $\mathcal{A}$ on X into a ring by taking:

- addition is the symmetric difference Δ: $U \Delta V = (U \setminus V) \cup (V \setminus U)$
- multiplication is $\cap$
- The additive identity is $\emptyset$ & A is its own additive inverse
- The multiplicative identity is X.

The most important example of a σ-algebra is the **Borel σ-algebra** on a topological space X, which is the σ-algebra $\mathcal{B}_X$ generated by the collection of open sets on X. Elements of $\mathcal{B}_X$ are called **Borel sets** on X.

See **Proposition 1.2** for equivalent ways to generate $\mathcal{B}_{\mathbb{R}}$.

Def: Let X be a set w/ σ-algebra $\mathcal{A}$. A **measure** on $(X, \mathcal{A})$ is a map $\mu : \mathcal{A} \to [0, \infty]$ s.t.

① $\mu(\emptyset) = 0$

② For $E_1, E_2, \ldots \in \mathcal{A}$ are pairwise disjoint, then $\mu(E_1 \cup E_2 \cup \ldots) = \mu(E_1) + \mu(E_2) + \cdots$

Then $(X, \mathcal{A})$ is a **measurable space** & elements of $\mathcal{A}$ are **measurable sets**. The triple $(X, \mathcal{A}, \mu)$ is a **measure space**.

The measure μ is **finite** if $\mu(X) < \infty$ (which obviously implies $\mu(E) < \infty \ \forall E \in \mathcal{A}$).

(In probability language, measurable sets are **events** & we always have $\mu(X) = 1$)

Note: ① is needed to rule out the degenerate case where $\mu(E) = \infty \ \forall E \in \mathcal{A}$.

Ex: (Counting measure) Let $\mathcal{A} = \mathcal{P}(X)$ & define $\mu(E) = |E|$. This is clearly a measure.

Ex: (Dirac measure) Let $\mathcal{A} = \mathcal{P}(X)$ & choose $x \in X$. Define $\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{else} \end{cases}$. Again, this is a measure.

Ex: Generalizing the previous 2 examples, if $\mathcal{A} = \mathcal{P}(X)$ & $f : X \to [0, \infty]$ is a function then

$$\mu_f(E) := \sum_{x \in E} f(x)$$

is a measure on X. Counting measure corresponds to the constant function $f(y) = 1 \ \forall y \in X$, & Dirac measure corresponds to the function $f(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{else} \end{cases}$.