Math 617: Day 19

Now we get Fubini's Theorem as an immediate consequence:

Fubini's Thm: Let $(X, \mathcal{M}_X, \mu_X)$ & $(Y, \mathcal{M}_Y, \mu_Y)$ be σ-finite measure spaces & let $f \in L^1(X\times Y, \mathcal{M}_X \otimes \mathcal{M}_Y, \mu_X \times \mu_Y)$. Then:

① $f_x \in L^1(Y, \mu_Y)$ for μ_X-a.e. $x \in X$ & $f^y \in L^1(X, \mu_X)$ for μ_Y-a.e. $y \in Y$.

② $x \mapsto \int_Y f(x,y)\, d\mu_Y(y)$ is in $L^1(X, \mu_X)$ for μ_Y-a.e. $y \in Y$ & $y \mapsto \int_X f(x,y)\, d\mu_X(x)$ is in $L^1(Y, \mu_Y)$ for μ_X-a.e. $x \in X$.

③ $\int_{X\times Y} f(x,y)\, d(\mu_X\times\mu_Y)(x,y) = \int_X \left(\int_Y f(x,y)\, d\mu_Y(y) \right) d\mu_X(x) = \int_Y \left(\int_X f(x,y)\, d\mu_X(x) \right) d\mu_Y(y)$.

Proof: By taking real & imaginary parts, we can assume f is real-valued. By taking positive & negative parts, we can also assume $f \geq 0$. But then the statement follows from Tonelli's Thm.

Remarks: ① The assumption $f \in L^1(X\times Y)$ is important... it's not enough even to have ① & ② (see HW).

② the theorem still holds (as does Tonelli's Thm) if we take the completions $\overline{\mathcal{M}_X \otimes \mathcal{M}_Y}$ & $\overline{\mu_X \times \mu_Y}$ of $\mathcal{M}_X \otimes \mathcal{M}_Y$ & $\mu_X \times \mu_Y$.

Higher-dimensional Lebesgue integrals

Df: *Lebesgue measure* m^n on $\mathbb{R}^n$ is the completion of $m \times m \times \dots \times m$ on $\mathcal{B}_{\mathbb{R}^n} = \mathcal{B}_\mathbb{R} \otimes \mathcal{B}_\mathbb{R} \otimes \dots \otimes \mathcal{B}_\mathbb{R}$ or, equivalently, on $\mathcal{L} \otimes \mathcal{L} \otimes \dots \otimes \mathcal{L}$.

The collection of *Lebesgue measurable* sets on $\mathbb{R}^n$ is denoted $\mathcal{L}^n$.

Essentially all the theorems that held for m also hold for m^n.

Prop: Lebesgue measure is translation-invariant on $\mathbb{R}^n$.

Prop: For $E \in \mathcal{L}^n$,

① $m^n(E) = \inf\{m^n(U) : U \supseteq E, U \text{ open}\} = \sup\{m^n(K) : K \subseteq E, K \text{ compact}\}$

② $E = A_1 \cup N_1 = A_2 \setminus N_2$, where A_1 is F_σ, A_2 is G_δ, & N_1, N_2 are null.

Of course, we now get the n-dimensional *Lebesgue integral* as the integral w.r.t. m^n. Of course, we then get Fubini's theorem for the work we've already done.

The next most important theorem about the Lebesgue integral in $\mathbb{R}^n$ is, of course, the change-of-variables formula:

Thm (Linear change of variables): Suppose $T:\mathbb{R}^n\to\mathbb{R}^n$ is an invertible linear transformation. If $f:\mathbb{R}^n\to\mathbb{C}$ is measurable, then so is $f\circ T$.

Moreover, if $f\in L^1(\mathbb{R}^n, m^n)$, then

$$\int f(x)\,dm^n(x) = |\det T|\int f\circ T(x)\,dm^n(x)$$

or, equivalently,

$$\int f(Tx)\,dm^n(x) = \frac{1}{|\det T|}\int f(x)\,dm^n(x)$$

Cor: If $E\in\mathcal{L}^n$, then $T(E)\in\mathcal{L}^n$ & $m^n(T(E)) = |\det T|\,m^n(E)$

Pf of Cor: Let $f=\chi_E$ & apply the change-of-variables theorem. ∎

Proof of Thm: The fact that $f\circ T$ is measurable follows from the same proof as in HW2 #4.

Also, if $T,S\in GL(n,\mathbb{R})$ & the result holds for T & S individually, then it holds for $T\circ S$:

$$\int f(x)\,dx = |\det T|\int f\circ T(x)\,dx = |\det T||\det S|\int (f\circ T)\circ S(x)\,dx = |\det(T\circ S)|\int f\circ(T\circ S)(x)\,dx.$$

So it suffices to prove the theorem for elementary linear transformations, which take the following 3 forms:

Row scale: $D_{k,c}(x_1,\dots,x_n) = (x_1,\dots,cx_k,\dots,x_n)$ $\quad c\neq 0$

Row add: $L_{j,k,c}(x_1,\dots,x_n) = (x_1,\dots,x_j+cx_k,\dots,x_n)$ $\quad j\neq k$

Row swap: $T_{j,k}(x_1,\dots,x_j,\dots,x_k,\dots,x_n) = (x_1,\dots,x_k,\dots,x_j,\dots,x_n)$

By Fubini's Thm, $\displaystyle\int f(x)\,dx = \int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}} f(x)\,dx_j\right)dm^{n-1} = \int_{\mathbb{R}^{n-1}} |c|\int f(x_1,\dots,cx_j,\dots,x_n)\,dx_j\Big)\,dm^{n-1} = \int_{\mathbb{R}^{n-1}} |\det D_{k,c}|\int_{\mathbb{R}} f\circ D_{k,c}(x)\,dx_j\Big)\,dm^{n-1}$

b/c $m(cE) = |c|\,m(E)$, so
this is true for simple functions &
hence, by taking limits, of all absolutely integrable functions

$$= |\det D_{k,c}|\int_{\mathbb{R}^n} f\circ D_{k,c}(x)\,dx$$

$$\int f(x)\,dx = \int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}} f(x)\,dx_j\right)dm^{n-1} = \int_{\mathbb{R}^{n-1}}\left(\int f(x_1,\dots,x_j+cx_k,\dots,x_n)\,dx_j\right)dm^{n-1} = \int_{\mathbb{R}^n} f\circ L_{j,k,c}(x)\,dx = |\det L_{j,k,c}|\int f\circ L_{j,k,c}(x)\,dx$$

$\det L_{j,k,c} = 1$

b/c $\int_{\mathbb{R}} g(x+a)\,dx = \int_{\mathbb{R}} g(x)\,dx$

$$\int f(x)\,dx = \int_{\mathbb{R}^{n-2}}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x)\,dx_j\right)dx_k\right)dm^{n-2} = \int_{\mathbb{R}^{n-2}}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f\circ T_{j,k}(x)\,dx_k\right)dx_j\right)dm^{n-2} = \int_{\mathbb{R}^n} f\circ T_{j,k}(x)\,dx = |\det T_{j,k}|\int_{\mathbb{R}^n} f\circ T_{j,k}(x)\,dx$$

$\det T_{j,k} = -1$

So we're done. ∎

Cor: Lebesgue measure is rotation-invariant. Since we already knew it was translation-invariant, this means it is invariant under all rigid motions (a.k.a. isometries)

Pf: If R is a rotation, $\det R = 1$. ∎

Of course, the really important change of variables theorem is for nonlinear changes of variables...