Math 617: Day 18

The goal now is to prove Tonelli's Thm & Fubini's Thm, which say that integrating over a product is the same as integrating one factor at a time (in either order). The key technical tool is the Monotone Class Lemma. First, a definition:

Def: A monotone class in X is a collection $\mathcal{B}$ of subsets of X satisfying the following closure properties:

① If $E_1 \subseteq E_2 \subseteq \dots$ is an increasing seq. of sets in $\mathcal{B}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{B}$.

② If $E_1 \supseteq E_2 \supseteq \dots$ is a decreasing seq. of sets in $\mathcal{B}$, then $\bigcap_{i=1}^{\infty} E_i \in \mathcal{B}$.

Monotone Class Lemma: Let $\mathcal{A}$ be an algebra on X. Then the σ-algebra $\langle \mathcal{A} \rangle$ generated by $\mathcal{A}$ is the smallest monotone class containing $\mathcal{A}$.

Tonelli's Thm: Let $(X, \mathcal{M}_X, \mu_X)$ & $(Y, \mathcal{M}_Y, \mu_Y)$ be σ-finite measure spaces & let $f \in L^+(X \times Y)$. Then

① The functions $x \mapsto \int_Y f(x,y)\, d\mu_Y(y)$ & $y \mapsto \int_X f(x,y)\, d\mu_X(x)$ are $\mathcal{M}_X$- & $\mathcal{M}_Y$-measurable, respectively.

② $$\int_{X \times Y} f(x,y)\, d(\mu_X \times \mu_Y)(x,y) = \int_X \left(\int_Y f(x,y)\, d\mu_Y(y) \right) d\mu_X(x) = \int_Y \left(\int_X f(x,y)\, d\mu_X(x) \right) d\mu_Y(y).$$

Pf: Since X is σ-finite, $X = \bigcup_{n=1}^{\infty} X_n$ w/ $\mu_X(X_n) < \infty$. But then, for any $\varphi \in L^+(X)$, $\varphi = \lim_{n\to\infty} \varphi \chi_{X_n}$, so the MCT implies

$$\int_X \varphi\, d\mu_X = \int \lim_{n\to\infty} \varphi \chi_{X_n}\, d\mu_X = \lim_{n\to\infty} \int \varphi \chi_{X_n}\, d\mu = \lim_{n\to\infty} \int_{X_n} \varphi\, d\mu_X$$

Similarly for Y, so the result will follow if we can prove it w/ X & Y replaced by X_n & Y_n, so we may as well assume $\mu_X(X) < \infty$ & $\mu_Y(Y) < \infty$, which implies $(\mu_X \times \mu_Y)(X \times Y) = \mu_X(X)\mu_Y(Y) < \infty$.

Now, since f is nonnegative & measurable, $f = \lim_{n\to\infty} f_n$, where the f_n are simple & $0 \le f_1 \le f_2 \le \dots \le f$.

Again, using the MCT it suffices to show that the results hold when we replace f by the simple function f_n.

Of course $f_n = \sum_{i=1}^{k} c_i \chi_{S_i}$ for $E_i \in \mathcal{M}_X \otimes \mathcal{M}_Y$. By linearity of the integral, it's enough to show that the result holds for a characteristic function χ_S w/ $S \in \mathcal{M}_X \otimes \mathcal{M}_Y$.

So we're now reduced to the case when $f = \chi_S$. Let $\mathcal{C}$ be the collection of all $S \in \mathcal{M}_X \otimes \mathcal{M}_Y$ for which the theorem is true when $f = \chi_S$.

Claim: $\mathcal{C}$ is a monotone class.

Proof of Claim: S'pose $S_1 \subseteq S_2 \subseteq \dots$ are in $\mathcal{C}$. Let $S = \bigcup_{i=1}^{\infty} S_i$. Then

$$\int_{X\times Y} \chi_S \, d(\mu_X\times\mu_Y) = \mu_X\times\mu_Y(S) \overset{\text{upward monotone convergence}}{=} \lim_{n\to\infty} \mu_X\times\mu_Y(S_n) = \lim_{n\to\infty} \int \chi_{S_n} \, d(\mu_X\times\mu_Y) = \lim_{n\to\infty} \int_X \left(\int_Y \chi_{S_n}(x,y)\, d\mu_Y(y)\right) d\mu_X(x)$$

$$\overset{\text{MCT}}{=} \int_X \left(\lim_{n\to\infty} \int_Y \chi_{S_n}(x,y)\, d\mu_Y(y)\right) d\mu_X(x)$$

$$\overset{\text{MCT}}{=} \int_X \left(\int_Y \lim_{n\to\infty} \chi_{S_n}(x,y)\, d\mu_Y(y)\right) d\mu_X(x)$$

$$= \int_X \left(\int_Y \chi_S(x,y)\, d\mu_Y(y)\right) d\mu_X(x)$$

and similarly for $\int_Y \left(\int_X \chi_S(x,y)\, d\mu_X(x)\right) d\mu_Y(y)$, so $S \in \mathcal{C}$.

OTOH, if $T_1 \supseteq T_2 \supseteq \dots$ are in $\mathcal{C}$, let $S_n = T_n^c$. Then for $T = \bigcap_{n=1}^{\infty} T_n$ & $S = \bigcup_{n=1}^{\infty} S_n$,

we know $T = S^c$ by deMorgan's Law. But now by the above argument we know

$$\mu_X\times\mu_Y(S) = \int \chi_S \, d(\mu_X\times\mu_Y) = \int_X \left(\int_Y \chi_S \, d\mu_Y(y)\right) d\mu_X(x)$$

so $$\int \chi_T \, d(\mu_X\times\mu_Y) = \mu_X\times\mu_Y(T) \overset{\text{by finiteness of } \mu_X\times\mu_Y}{=} \mu_X\times\mu_Y(X\times Y) - \mu_X\times\mu_Y(S) = \int 1 \, d(\mu_X\times\mu_Y) - \int \chi_S \, d(\mu_X\times\mu_Y)$$

$$= \int_X \left(\int_Y 1 \, d\mu_Y(y)\right) d\mu_X(x) - \int_X \left(\int_Y \chi_S(x,y)\, d\mu_Y(y)\right) d\mu_X(x)$$

$$= \int_X \left(\int_Y (1 - \chi_S(x,y))\, d\mu_Y(y)\right) d\mu_X(x)$$

$$= \int_X \left(\int_Y \chi_T(x,y)\, d\mu_Y(y)\right) d\mu_X(x),$$

so $T \in \mathcal{C}$ & $\mathcal{C}$ is a monotone class. ☒

But now, if $S = E\times F$, then $S \in \mathcal{C}$, so we see that $\mathcal{C} \supseteq \mathcal{B}_0$ from the proof of the existence of product measure.

Then the Monotone Class Lemma implies that $\mathcal{C}$ contains $\langle \mathcal{B}_0 \rangle = \mathcal{M}_X \otimes \mathcal{M}_Y$, and so the theorem is true $\forall\, S \in \mathcal{M}_X \otimes \mathcal{M}_Y$ &

we're done. ☒