Math 617: Day 17

Now, the point is to define the product measure on a product measurable space. This turns out to be much easier in the case of σ-finite measures:

Def: A measure space $(X, \mathcal{M}, \mu)$ is σ-finite if $X = \bigcup_{i=1}^{\infty} E_i$ where $\mu(E_i) < \infty$ $\forall\, i \in \mathbb{N}$.

Ex: $\cdot$ obviously, any finite measure space is σ-finite.

$\cdot$ Also, $(\mathbb{R}, \mathcal{L}, m)$ is σ-finite since $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$ & $m([-n,n]) = 2n < \infty$ $\forall\, n \in \mathbb{N}$.

Thm (Existence & Uniqueness of Product Measure): Let $(X, \mathcal{M}_X, \mu_X)$ & $(Y, \mathcal{M}_Y, \mu_Y)$ be σ-finite measurable spaces. Then there exist a unique measure $\mu_X \times \mu_Y$ on $(X \times Y, \mathcal{M}_X \otimes \mathcal{M}_Y)$ s.t.

$$(\mu_X \times \mu_Y)(E \times F) = \mu_X(E)\,\mu_Y(F) \quad \forall\, E \in \mathcal{M}_X,\ F \in \mathcal{M}_Y.$$

Proof: The idea is to use the Hahn-Kolmogorov theorem. Just as Lebesgue measure was the measure generated by the elementary pre-measure on $\mathbb{R}$, we want to define an "elementary product pre-measure", show it's really a pre-measure, & then feed it to Hahn-Kolmogorov.

Let $\mathcal{B}_0$ be the collection of all finite unions $\bigcup_{i=1}^{n} E_i \times F_i$ where $E_i \in \mathcal{M}_X$, $F_i \in \mathcal{M}_Y$. Then $\mathcal{B}_0$ is obviously closed under finite unions & since $(E \times F)^c = (X \times F^c) \cup (E^c \times Y)$, taking complements. So $\mathcal{B}_0$ is an algebra.

Now, for any $S \in \mathcal{B}_0$, we can write S as a disjoint union $S = \bigsqcup_{i=1}^{n} E_i \times F_i$ & we define

$$\mu_0(S) := \sum_{i=1}^{n} \mu_X(E_i)\,\mu_Y(F_i),$$

which is obviously finitely additive.

The goal is to show that μ_0 is a premeasure. Obviously $\mu_0(\emptyset) = \mu_0(\emptyset \times \emptyset) = \mu_X(\emptyset)\mu_Y(\emptyset) = 0 \cdot 0 = 0$, so it suffices to show that if

$$S = \bigcup_{i=1}^{\infty} S_i \text{ is in } \mathcal{B}_0 \text{ with each } S_i \in \mathcal{B}_0 \text{ and the } S_i \text{ disjoint, then } \mu_0(S) = \sum_{i=1}^{\infty} \mu_0(S_i).$$

Using finite additivity, we can reduce to the case when $S = E \times F$ & $S_i = E_i \times F_i$ for each i, so then the goal is to show

$$\mu_X(E)\,\mu_Y(F) = \sum_{i=1}^{\infty} \mu_X(E_i)\,\mu_Y(F_i).$$

Now, the trick is going to be to see that we can get this equation by integrating both sides of

$$\chi_E(x)\,\chi_F(y) = \sum_{i=1}^{\infty} \chi_{E_i}(x)\,\chi_{F_i}(y)$$

w.r.t. x & y. Note, first of all, that the above equation is true $\forall\, x \in X, y \in Y$. Now, first fix x & integrate w.r.t. y:

$$\int_Y \chi_E(x)\,\chi_F(y)\, d\mu_Y(y) = \int_Y \sum_{i=1}^{\infty} \chi_{E_i}(x)\,\chi_{F_i}(y)\, d\mu_Y(y)$$

The LHS is just $\chi_E(x)\,\mu_Y(F)$. For the RHS, note that

by Monotone Convergence Thm

$$\int_Y \sum_{i=1}^{\infty} \chi_{E_i}(x)\chi_{F_i}(y)\, d\mu_Y(y) = \int_Y \lim_{n\to\infty} \sum_{i=1}^{n} \chi_{E_i}(x)\chi_{F_i}(y)\, d\mu_Y(y) = \lim_{n\to\infty} \int_Y \sum_{i=1}^{n} \chi_{E_i}(x)\chi_{F_i}(y)\, d\mu_Y(y)$$

by finite additivity of integral

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \int_Y \chi_{E_i}(x)\chi_{F_i}(y)\, d\mu_Y(y)$$

$$= \sum_{i=1}^{\infty} \chi_{E_i}(x)\,\mu_Y(F_i).$$

So, putting this together, we see that

$$\chi_E(x)\,\mu_Y(F) = \sum_{i=1}^{\infty} \chi_{E_i}(x)\,\mu_Y(F_i).$$

Of course, the same thing happens when we integrate w.r.t. x:

$$\mu_X(E)\mu_Y(F) = \int_X \chi_E(x)\mu_Y(F)\, d\mu_X(x) = \int_X \sum_{i=1}^{\infty} \chi_{E_i}(x)\mu_Y(F_i)\, d\mu_X(x) = \dots = \sum_{i=1}^{\infty} \mu_X(E_i)\mu_Y(F_i).$$

So now μ_0 is a pre-measure & we can apply Hahn-Kolmogorov to get a unique measure $\mu_X \times \mu_Y$ which extends μ_0 & hence satisfies the conditions in the theorem. ∎