Math 617: Day 16

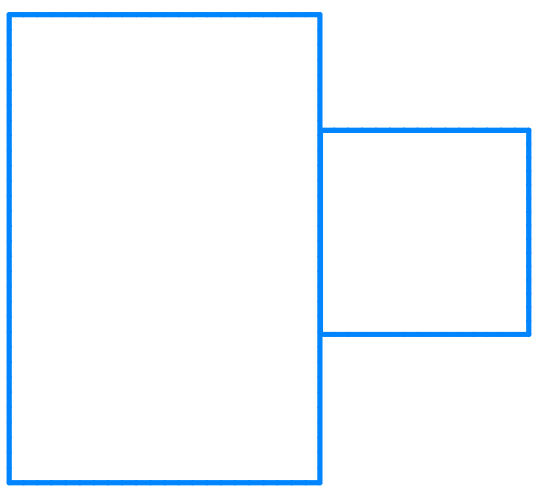
Product Measures

The next goal is to extend the Lebesgue integral to higher dimensions. We already have the machinery to define integration on an arbitrary measure space, so the real challenge is to figure out what Lebesgue measure is supposed to mean in, say, $\mathbb{R}^d$. Of course, the idea is to define it to be the product of Lebesgue measure on each $\mathbb{R}$ factor.

In general, given sets X & Y, we have their Cartesian product $X \times Y = \{(x,y) : x \in X, y \in Y\}$ & the associated projections $\pi_X : X \times Y \to X$ & $\pi_Y : X \times Y \to Y$ given by $\pi_X((x,y)) = x$ & $\pi_Y((x,y)) = y$.

Now, if $(X, \mathcal{M}_X)$ & $(Y, \mathcal{M}_Y)$ are measurable spaces, we would like to find a measure structure on the product $X \times Y$. Unfortunately, we can't just take $(X \times Y, \mathcal{M}_X \times \mathcal{M}_Y)$... after all, $\mathcal{M}_X \times \mathcal{M}_Y = \{(E,F) : E \in \mathcal{M}_X, F \in \mathcal{M}_Y\}$ is not a collection of subsets of $X \times Y$.

Even $\{E \times F : E \in \mathcal{M}_X, F \in \mathcal{M}_Y\}$ isn't quite right because this is not a σ-algebra: if $X = Y = \mathbb{R}$ & $\mathcal{M}_X = \mathcal{M}_Y = \mathcal{B}_{\mathbb{R}}$, then we can take a union of boxes & get something which is not a product:



Of course, the right thing to do is to define $\mathcal{M}_X \otimes \mathcal{M}_Y$ to be the σ-algebra generated by the sets $E \times F$ w/ $E \in \mathcal{M}_X$ & $F \in \mathcal{M}_Y$.

Equivalently,

$$\mathcal{M}_X \otimes \mathcal{M}_Y = \langle \pi_X^*(\mathcal{M}_X) \cup \pi_Y^*(\mathcal{M}_Y) \rangle,$$

where

$$\pi_X^*(\mathcal{M}_X) = \{\pi_X^{-1}(E) : E \in \mathcal{M}_X\} = \{E \times Y : E \in \mathcal{M}_X\}$$

$$\pi_Y^*(\mathcal{M}_Y) = \{\pi_Y^{-1}(F) : F \in \mathcal{M}_Y\} = \{X \times F : F \in \mathcal{M}_Y\}$$

Def: Given a set $E \subseteq X \times Y$, the **sections** E_x & E^y are

$$E_x := \{y \in Y : (x,y) \in E\} \quad \& \quad E^y := \{x \in X : (x,y) \in E\}$$

Also, given a function $f : X \times Y \to \mathbb{R}$, we have the corresponding sections of f:

$f_x : Y \to \mathbb{R}$ given by $f_x(y) = f(x,y)$

$f^y : X \to \mathbb{R}$ given by $f^y(x) = f(x,y)$.

Prop: If $(X, \mathcal{M}_X)$ & $(Y, \mathcal{M}_Y)$ are measurable spaces, then

① For $E \in \mathcal{M}_X \otimes \mathcal{M}_Y$, $E_x \in \mathcal{M}_Y$ $\forall x \in X$ & $E^y \in \mathcal{M}_X$ $\forall y \in Y$.

② If $f : X \times Y \to \mathbb{R}$ is measurable, then $f_x : Y \to \mathbb{R}$ is $\mathcal{M}_Y$-measurable $\forall x \in X$ & $f^y : X \to \mathbb{R}$ is $\mathcal{M}_X$-measurable $\forall y \in Y$.