Math 617: Day 15

To get a handle on the different notions of convergence from last time, let's look at some examples of sequences, all of which converge to 0 in some sense(s), but don't in others:

Ex: Consider the sequences

$$f_n = \chi_{[n,n+1]}, \qquad g_n = \frac{1}{n}\chi_{[0,n]}, \qquad h_n = n\,\chi_{[\frac{1}{n},\frac{2}{n}]}, \qquad p_n = \chi_{\left[\frac{n-2^k}{2^k},\, \frac{n-2^k+1}{2^k}\right]} \text{ for } 2^k \le n < 2^{k+1}.$$

Each of these converges to 0 in some sense... but which?

A: $f_n \to 0$ pointwise & pointwise a.e., not uniformly, in L^∞ norm, almost uniformly, in L^1 norm, or in measure

$g_n \to 0$ uniformly ($\Rightarrow$ pointwise, pointwise a.e., in L^∞ norm, almost uniformly, in measure), but not in L^1 norm.

$h_n \to 0$ pointwise & pointwise a.e. and almost uniformly & in measure, but not uniformly, in L^∞ norm, or in L^1 norm.

$p_n \to 0$ in measure & in L^1 norm, but not pointwise a.e. ($\Rightarrow$ not pointwise, not uniformly, not in L^∞ norm, not almost uniformly).

Prop: Let $(X, \mathcal{M}, \mu)$ be a measure space & let $f_n : X \to \mathbb{C}$, $f : X \to \mathbb{C}$ be measurable functions.

(i) If $f_n \to f$ uniformly, then $f_n \to f$ pointwise

(ii) If $f_n \to f$ uniformly, then $f_n \to f$ in L^∞ norm. Conversely, if $f_n \to f$ in L^∞ norm, then $f_n \to f$ outside a null set

(iii) If $f_n \to f$ in L^∞ norm, then $f_n \to f$ almost uniformly.

(iv) If $f_n \to f$ almost uniformly, then $f_n \to f$ pointwise almost everywhere

(v) If $f_n \to f$ pointwise, then $f_n \to f$ pointwise almost everywhere

(vi) If $f_n \to f$ in L^1 norm, then $f_n \to f$ in measure

(vii) If $f_n \to f$ almost uniformly, then $f_n \to f$ in measure

Or, using the numbers from last time, we have the following graph of implications:

$$\begin{tikzcd}
2 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & 1 \arrow[dr, Rightarrow] & \\
4 \arrow[r, Rightarrow] & 5 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & 3 \\
 & 7 & \\
 & 6 \arrow[u, Rightarrow] &
\end{tikzcd}$$

Proof of (vi): Let $\varepsilon > 0$ & let $E_{n,\varepsilon} = \{x \in X : |f_n(x) - f(x)| \ge \varepsilon\}$. The goal is to show that $\mu(E_{n,\varepsilon}) \to 0$ as $n \to \infty$.

So let $\alpha > 0$. Since $f_n \to f$ in L^1 norm, we know $\|f_n - f\|_{L^1} = \int |f_n - f|\,d\mu \to 0$ as $n \to \infty$, so $\exists\, N \in \mathbb{N}$ s.t.

$n \ge N$ implies $\int |f_n - f|\,d\mu < \varepsilon\alpha$. In particular,

$$\varepsilon\,\mu(E_{n,\varepsilon}) = \int_{E_{n,\varepsilon}} \varepsilon\,d\mu \le \int_{E_{n,\varepsilon}} |f_n - f|\,d\mu < \int |f_n - f|\,d\mu < \varepsilon\alpha,$$

So we see that $\mu(E_{n,\varepsilon}) < \alpha$. But since the choice of $\alpha > 0$ was arbitrary, we see that $\mu(E_{n,\varepsilon}) \to 0$, as desired. ∎

Egorov's Thm: Let X have finite measure & suppose $f_n : X \to \mathbb{C}$ & $f : X \to \mathbb{C}$ are measurable. Then

$$f_n \to f \text{ pointwise a.e.} \iff f_n \to f \text{ almost uniformly}.$$

Note: obviously, the finite measure condition is necessary, as our example $f_n = \chi_{[n, n+1]}$ shows.

Proof: We already know the $\Leftarrow$ direction, so we just need to show the $\Rightarrow$ direction.

So assume $f_n \to f$ pointwise a.e. & let $\varepsilon > 0$. The goal is to show $\exists\ E \subseteq X$ s.t. $\mu(E) \leq \varepsilon$ & $f_n \to f$ uniformly on $X \setminus E$.

For each n & k, define

$$E_{n,k} = \bigcup_{m=n}^{\infty} \{x \in X : |f_m(x) - f(x)| \geq \tfrac{1}{k}\}$$

Then certainly $E_{1,k} \supseteq E_{2,k} \supseteq \ldots$ &

$$\bigcap_{n=1}^{\infty} E_{n,k} = \{x : |f_m(x) - f(x)| \geq 1/k \ \ \forall m \in \mathbb{N}\} = \{x : f_n \not\to f\},$$ which has measure 0.

Now, since $\mu(X) < \infty$, all $\mu(E_{n,k}) < \infty$, so we can apply downward monotone convergence to get

$$0 = \mu\left(\bigcap_{n=1}^{\infty} E_{n,k}\right) = \lim_{n \to \infty} \mu(E_{n,k}).$$

In particular, for each k $\exists\ n_k$ s.t. $\mu(E_{n_k,k}) < \frac{\varepsilon}{2^k}$.

Define $E = \bigcup_{k=1}^{\infty} E_{n_k,k}$. Notice that, for any k, $\{x : |f_n(x) - f(x)| < \frac{1}{k} \text{ for all } n \geq n_k\} = X \setminus E_{n_k,k} \subseteq X \setminus E$.

Then for any $\alpha > 0$ $\exists\ k \geq \frac{1}{\alpha}$, so $n \geq n_k$ implies

$$|f_n(x) - f(x)| < \tfrac{1}{k} \leq \alpha$$

on $X \setminus E$. Of course, α was arbitrary, so we see that $f_n \to f$ uniformly on $X \setminus E$. But now

$$\mu(E) \leq \sum \mu(E_{n_k,k}) \leq \sum \varepsilon/2^k = \varepsilon,$$

so we're done. ∎