Math 617: Day 14

Dominated Convergence Thm: Let $(X,\mathcal{M},\mu)$ be a measure space & let $f_1, f_2, \ldots : X \to \mathbb{C}$ be a sequence of integrable functions s.t.

(i) $\exists$ measurable $f: X \to \mathbb{C}$ s.t. $f_n \to f$ μ-a.e.

(ii) $\exists$ nonnegative $g \in L^1(X,\mathcal{M},\mu)$ s.t. $|f_n| \leq g$ μ-a.e. $\forall n \in \mathbb{N}$.

Then $f \in L^1(X,\mathcal{M},\mu)$ &

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu.$$

Q: How can this fail if there is no such g?

A: Consider $f_n(x) = \chi_{[n,n+1]} = \begin{cases} 1 & \text{if } x \in [n,n+1] \\ 0 & \text{else} \end{cases}$

Then the f_n are simple functions & hence obviously integrable, & $f_n \to 0$ pointwise. But obviously

$$\int f_n\,d\mu = 1 \quad \forall n, \text{ so } \lim_{n\to\infty} \int f_n\,d\mu = 1 \neq 0.$$

Note that this example shows that the inequality in Fatou's Lemma can't be an equality. Essentially, by pushing it off to infinity, you can destroy mass in the limit, but Fatou's Lemma says you cannot create mass in the limit.

Proof of DCT: We may as well modify the f_n & f so that $f_n \to f$ pointwise (rather than μ-a.e.) & $|f_n| \leq g$ everywhere. Also, by dealing with real & imaginary parts separately, we might as well assume the f_n & f are real-valued, so $-g \leq f_n \leq g$, which obviously implies $-g \leq f \leq g$.

We don't yet know if $\lim_{n\to\infty} \int f_n\,d\mu$ exists, but of course $\liminf_{n\to\infty} \int f_n\,d\mu$ & $\limsup_{n\to\infty} \int f_n\,d\mu$ do exist, & the result will follow if we can show that $\limsup_{n\to\infty} \int f_n\,d\mu \leq \int f\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu$.

Now, $-g \leq f_n \leq g$ implies that $f_n + g \geq 0$ & $g - f_n \geq 0$, so these are 2 sequences of nonnegative functions.

Q: What can we do w/ these 2 sequences?

A: Apply Fatou's lemma!

$$\int f\,d\mu + \int g\,d\mu = \int (f+g)\,d\mu = \int \lim_{n\to\infty} (f_n+g)\,d\mu \leq \liminf_{n\to\infty} \int (f_n+g)\,d\mu = \liminf_{n\to\infty} \int f_n\,d\mu + \int g\,d\mu$$

so $\displaystyle\int f\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu$

and

$$\int g\,d\mu - \int f\,d\mu = \int (g-f)\,d\mu = \int \lim_{n\to\infty} (g-f_n)\,d\mu \le \liminf_{n\to\infty} \int (g-f_n)\,d\mu = \int g\,d\mu - \limsup_{n\to\infty} \int f_n\,d\mu$$

So $-\int f\,d\mu \le -\limsup_{n\to\infty} \int f_n\,d\mu$ or $\limsup_{n\to\infty} \int f_n\,d\mu \le \int f\,d\mu$.

So combining these gives

$$\limsup_{n\to\infty} \int f_n\,d\mu \le \int f\,d\mu \le \liminf_{n\to\infty} \int f_n\,d\mu,$$

completing the proof. ▨

When $X=\mathbb{R}$, the integral we're talking about is called the Lebesgue integral.

Prop: If $f:[a,b]\to\mathbb{R}$ is bdd, then:

① If f is Riemann integrable, then f is Lebesgue integrable & $\int_a^b f(x)\,dx = \int_{[a,b]} f\,dm$.

② f is Riemann integrable $\Leftrightarrow$ $m(\{x : f \text{ discontinuous at } x\}) = 0$.

① is hopefully obvious, given how we defined the Lebesgue integral. ② is harder, but hopefully intuitively sort of clear.

Types of Convergence:

We know from classical analysis about pointwise convergence & uniform convergence, and hopefully remember examples like $f_n(x) = \frac{x}{n}$, which converges pointwise but not uniformly to 0.

But we can extend the list with many other types of convergence:

① f_n converges to f pointwise
② f_n converges to f uniformly
③ f_n converges to f pointwise almost everywhere if $f_n(x)\to f(x)$ for μ-a.e. $x\in X$.
④ f_n converges to f uniformly almost everywhere (or essentially uniformly or in L^∞ norm) if $\forall \varepsilon>0\ \exists N\in\mathbb{N}$ s.t. $n\ge N \Rightarrow |f_n(x)-f(x)|\le\varepsilon$ μ-a.e.
⑤ f_n converges to f almost uniformly if $\forall \varepsilon>0\ \exists E\in\mathcal{M}$ w/ $\mu(E)\le\varepsilon$ s.t. f_n converges uniformly to f in E^c.
⑥ f_n converges to f in L^1 norm if $\|f_n-f\|_{L^1} = \int |f_n-f|\,d\mu \to 0$ as $n\to\infty$.
⑦ f_n converges to f in measure if $\forall\varepsilon>0$, $\mu(\{x\in X : |f_n(x)-f(x)|\ge\varepsilon\})\to 0$ as $n\to\infty$.

Note: In probability theory (where f_n & f are random variables), ⑥ is called convergence in mean, ③ is called almost sure convergence,

and ⑦ is called convergence in probability.