Math 617: Day 13

So far we've only defined the integral for nonnegative functions, but the extension to arbitrary real-valued functions is easy:

Recall that, if $f: X \to \mathbb{R}$, then $f^+(x) = \max(f(x), 0)$ & $f^-(x) = \max(-f(x), 0)$, so that

$$f(x) = f^+(x) - f^-(x) \qquad \forall x \in X.$$

Then, for $(X, \mathcal{M}, \mu)$ a measure space & $f: X \to \mathbb{R}$ absolutely integrable, meaning $\|f\|_{L^1(X, \mathcal{B}, \mu)} := \int |f| \, d\mu < \infty$, we define

$$\int f \, d\mu := \int f^+ \, d\mu - \int f^- \, d\mu.$$

Similarly, if $f: X \to \mathbb{C}$ is absolutely integrable, define

$$\int f \, d\mu := \int \mathrm{Re}(f) \, d\mu + i \int \mathrm{Im}(f) \, d\mu.$$

Prop: For absolutely integrable $f, g: X \to \mathbb{C}$,

$$|f-g| = 0 \ \mu\text{-a.e.} \iff f = g \ \mu\text{-a.e.}$$

and either implies that

$$\int f \, d\mu = \int g \, d\mu$$

Proof: We know from before that $|f-g| = 0$ μ-a.e. $\iff \int |f-g| \, d\mu = 0$. Since $|h| = h^+ + h^-$ for any function h, this is equivalent to $\int (f-g)^+ d\mu + \int (f-g)^- d\mu = 0$. Of course, $\int (f-g)^+ d\mu \geq 0$ & $\int (f-g)^- d\mu \geq 0$, so this is true $\iff$ $\int (f-g)^+ d\mu = 0$ & $\int (f-g)^- d\mu = 0$. In turn, this is true $\iff$ $\max(f-g, 0) = 0$ μ-a.e. & $\max(-(f-g), 0) = 0$ μ-a.e.

But clearly this is equivalent to $f = g$ μ-a.e., so we have the $\iff$. In turn, $f = g$ μ-a.e. $\Rightarrow$ $f^+ = g^+$ μ-a.e. & $f^- = g^-$ μ-a.e., so then $\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu = \int g^+ d\mu - \int g^- d\mu = \int g \, d\mu$. ☒

Now, we want to turn the collection of integrable functions into a Banach space w/ norm $\|f\|_{L^1} := \int |f| \, d\mu$. The issue is that, as above, we can have $0 = \|f-g\|_{L^1} = \int |f-g| \, d\mu$ even when $f \neq g$ so long as $f = g$ μ-a.e.

We resolve this issue by defining $L^1(X,\mathcal{M},\mu)$ to be the set of equivalence classes of integrable functions on X, where $f\sim g$ if $f=g$ μ-a.e.

(and, in fact, we allow $f\in L^1(X,\mathcal{M},\mu)$ to be defined on a subset of X so long as the set on which it's undefined has measure 0.

With this definition, we have:

Prop: Let $(X,\mathcal{M},\mu)$ be a measure space. Then $L^1(X,\mathcal{M},\mu)$ is a normed complex vector space

Proof: If $f,g\in L^1(X,\mathcal{M},\mu)$ & $c\in\mathbb{C}$, then

$$\int|cf+g|\,d\mu\leq\int|cf|\,d\mu+\int|g|\,d\mu=|c|\int|f|\,d\mu+\int|g|\,d\mu<\infty,$$

so $cf+g\in L^1(X,\mathcal{M},\mu)$ & this is a complex vector space.

Implicitly we just showed that the L^1 norm $\|\cdot\|_{L^1}$ satisfies the triangle inequality $\|f+g\|_{L^1}\leq\|f\|_{L^1}+\|g\|_{L^1}$ & homogeneity $\|cf\|_{L^1}=|c|\,\|f\|_{L^1}$

and the point of taking equivalence classes was to get $\|f-g\|_{L^1}=0\Leftrightarrow f=g$ (where here "f" & "g" are really equivalence classes.

So this is a normed vector space. ∎

In fact, as we will see shortly, $L^1(X,\mathcal{M},\mu)$ is a complete normed vector space, otherwise known as a Banach space.

Prop: The map $I:L^1(X,\mathcal{M},\mu)\to\mathbb{C}$ given by $I(f)=\int f\,d\mu$ is a linear map.

In other words, integration is a linear functional on $L^1(X,\mathcal{M},\mu)$.

Dominated Convergence Theorem: Let $(X,\mathcal{M},\mu)$ be a measure space & let $f_1,f_2,\ldots:X\to\mathbb{C}$ be a sequence of integrable functions s.t.

(i) $\exists$ measurable $f:X\to\mathbb{C}$ s.t. $f_n\to f$ μ-a.e.

(ii) $\exists$ nonnegative $g\in L^1(X,\mathcal{M},\mu)$ s.t. $|f_n|\leq g$ μ-a.e. $\forall n\in\mathbb{N}$.

Then $f\in L^1(X,\mathcal{M},\mu)$ &

$$\int f\,d\mu=\lim_{n\to\infty}\int f_n\,d\mu.$$

Q: How can this fail if there is no such g?

A: Consider $f_n(x)=\chi_{[n,n+1]}=\begin{cases}1 & \text{if } x\in[n,n+1]\\ 0 & \text{else}\end{cases}$

Then the f_n are simple functions & hence obviously integrable, & $f_n\to 0$ pointwise. But obviously

$$\int f_n\,d\mu=1\quad\forall n,\text{ so }\lim_{n\to\infty}\int f_n\,d\mu=1\neq 0.$$