Math 617: Day 12

Now, of course, we extend the integral to all of L^+ in the obvious way: for $f \in L^+$, define

$$\int f \, d\mu := \sup\left\{ \int \varphi \, d\mu : 0 \le \varphi \le f \text{ is simple} \right\}$$

Now, by the earlier theorem we know any $f \in L^+$ is the pointwise limit of an increasing sequence of simple functions: $f = \lim_{n\to\infty} f_n$ where $0 \le f_1 \le f_2 \le \dots \le f$ are simple. The obvious theorem you would want to be true is that $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$. This turns out to be true:

Monotone Convergence Thm: If $f_1, f_2, \dots \in L^+$ is an increasing sequence (i.e., $f_n \le f_{n+1}$ $\forall n \in \mathbb{N}$) & $f = \lim_{n\to\infty} f_n$, then

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

__Remark:__ Unlike in the case of the Riemann integral, we don't have to assume that the limit f is integrable... we already know that it is!

__Proof:__ Since $\left(\int f_n \, d\mu\right)_{n=1}^{\infty}$ is an increasing sequence, it has a (possibly infinite) limit. Moreover, since $f_n \le f$ $\forall n$, we know

$$\lim_{n\to\infty} \int f_n \, d\mu \le \int f \, d\mu.$$

To prove the reverse inequality, it suffices to show $\int g \, d\mu \le \lim_{n\to\infty} \int f_n \, d\mu$ for any simple $0 \le g \le f$.

So choose an arbitrary such g.

Now, let $0 < \varepsilon < 1$ & consider the sets

$$E_n = \{x \in X : f_n(x) > \varepsilon g(x)\}$$

Since $f_1 \le f_2 \le \dots$, we certainly have $E_1 \subseteq E_2 \subseteq \dots$. Also, since $g \le f = \lim_{n\to\infty} f_n = \sup_n f_n$, we know that

$$X = \bigcup_{i=1}^{\infty} E_i.$$

Now, $\int f_n \, d\mu \ge \int_{E_n} f_n \, d\mu \ge \int_{E_n} \varepsilon g \, d\mu = \varepsilon \int_{E_n} g \, d\mu$.

Taking the limit as $n \to \infty$, we see $\lim_{n\to\infty} \int f_n \, d\mu \ge \varepsilon \lim_{n\to\infty} \int_{E_n} g \, d\mu = \varepsilon \lim_{n\to\infty} \mu_g(E_n)$, where μ_g is the measure from last time. Since μ_g is a measure, it satisfies upper monotone convergence, so

$$\lim_{n\to\infty} \mu_g(E_n) = \mu_g\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu_g(X) = \int g \, d\mu.$$

So now we have $\lim_{n\to\infty} \int f_n \, d\mu \ge \varepsilon \int g \, d\mu$ for any $0 < \varepsilon < 1$.

But then letting $\varepsilon \to 1$ gives

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \int g \, d\mu,$$

as desired. ▨

Remark: of course, the result still holds if we only assume $f_n \leq f_{n+1}$ μ-a.e. for each n.

Tonelli's Thm for Sums & Integrals: Let $(X, \mathcal{M}, \mu)$ be a measure space & $f_1, f_2, \ldots \in L^+$. Then

$$\int \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \left(\int f_n \, d\mu \right)$$

Proof: Apply monotone convergence to the sequence of partial sums. ▨

Of course, the results of the Monotone Convergence Theorem can fail if the sequence isn't monotone, but we do still get an inequality:

Fatou's Lemma: Let $(X, \mathcal{M}, \mu)$ be a measure space & let $f_1, f_2, \ldots \in L^+$. Then

$$\int \left(\liminf_{n\to\infty} f_n \right) d\mu \leq \liminf_{n\to\infty} \left(\int f_n \, d\mu \right)$$

Proof: Let $F_N = \inf_{n \geq N} f_n$. By definition of $\liminf$, we have $\liminf_{n\to\infty} f_n = \lim_{N\to\infty} F_N$.

Now, the F_N are measurable & nondecreasing, so the Monotone Convergence Thm implies that

(*) $$\int \left(\liminf_{n\to\infty} f_n \right) d\mu = \int \left(\lim_{N\to\infty} F_N \right) d\mu = \lim_{N\to\infty} \int F_N \, d\mu.$$

On the other hand, $F_N \leq f_n$ $\forall\, n \geq N$, so $\int F_N \, d\mu \leq \int f_n \, d\mu$ $\forall\, n \geq N$ & hence

$$\int F_N \, d\mu \leq \inf_{n \geq N} \int f_n \, d\mu.$$

Combining with (*), then, gives that

$$\int \left(\liminf_{n\to\infty} f_n \right) d\mu = \lim_{N\to\infty} \int F_N \, d\mu \leq \lim_{N\to\infty} \inf_{n \geq N} \int f_n \, d\mu = \liminf_{n\to\infty} \int f_n \, d\mu,$$

as desired. ▨