Math 617: Day 11

Df: A function $f: X \to \mathbb{C}$ is **simple** if f is measurable & the range of f is finite.

Of course, any simple function f whose range is $\{z_1, \ldots, z_n\}$ can be written as a finite linear combination of indicator functions:

$$f = \sum_{i=1}^{n} z_i \chi_{f^{-1}(z_i)}$$ where the $f^{-1}(z_i)$ are, of course, measurable sets.

The simple functions play the same role that the piecewise constant functions did in the Riemann integral. Indeed, the key point is that any measurable function can be approximated by simple functions (just as any continuous [or, really, Riemann integrable] function can be approximated by piecewise constant functions):

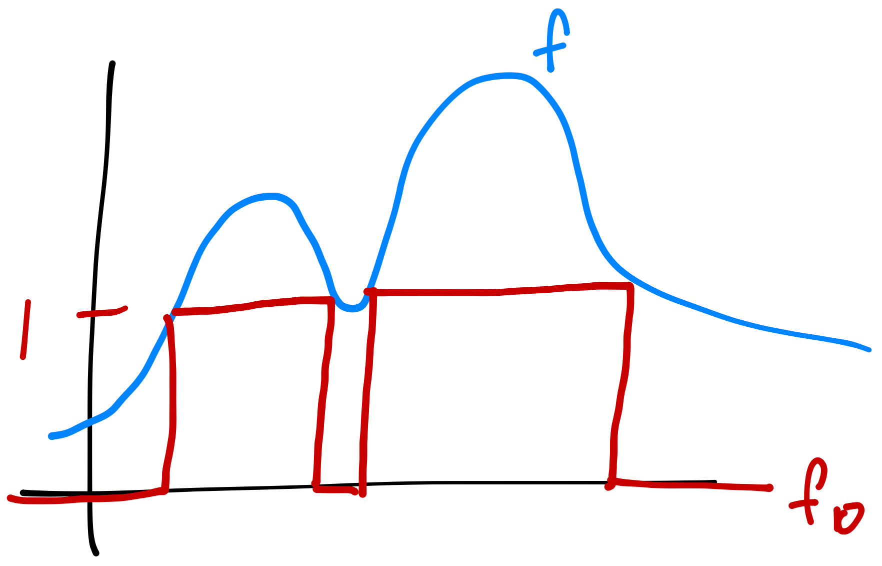
Thm: Let $(X, \mathcal{M})$ be a measurable space. If $f: [0,\infty]$ is measurable, then f is the pointwise limit of a sequence of simple functions.

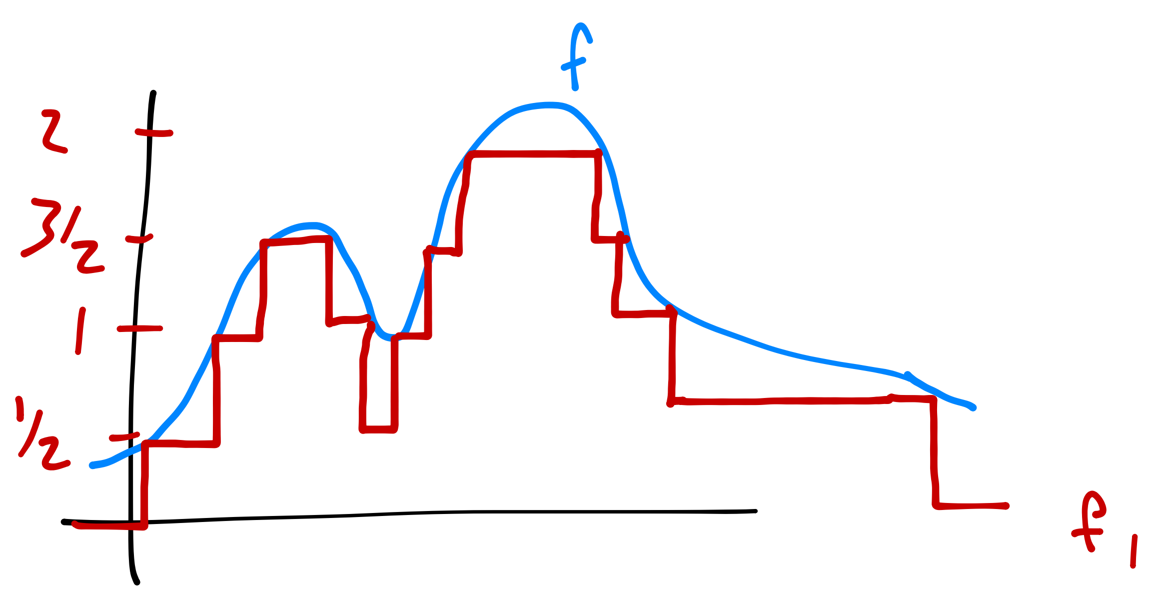
More precisely, $\exists$ a sequence $\{f_n\}$ of simple functions s.t. $0 \leq f_1 \leq f_2 \leq \ldots \leq f$, $f(x) = \lim_{n\to\infty} f_n(x)$ $\forall x \in X$, and $f_n \to f$ uniformly on any set where f is bounded.

Proof: For each n, let f_n be given by

$$f_n = \sum_{k=0}^{n\cdot 2^n} \frac{k}{2^n} \chi_{f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right)}$$

Another way of saying this is that
$$f_n(x) = \begin{cases} \dfrac{\lfloor 2^n \min(f(x), n) \rfloor}{2^n} & \text{if } |x| \leq n \\ 0 & \text{else} \end{cases}$$





Now, the point is to define the integral in the obvious way: if $(X, \mathcal{M}, \mu)$ is a measure space & L^+ is the space of nonnegative measurable functions & $f \in L^+$ is simple, then $f = \sum_1^n c_i \chi_{E_i}$ & we define the **integral**

$$\int f \, d\mu = \sum_1^n c_i \mu(E_i).$$

Similarly, if $A \subseteq X$ is measurable, then $\int_A f\,d\mu := \int f \chi_A \,d\mu$.

The following things hold:

Prop: For $f, g \in L^+$ simple:

1. $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$
2. $\int cf\,d\mu = c\int f\,d\mu \quad \forall\, c \in [0,\infty]$
3. $\int f\,d\mu < \infty \iff f$ is finite almost everywhere (i.e. $\{x \in X : f(x) = \infty\}$ has measure 0)
4. $\int f\,d\mu = 0 \iff f = 0$ almost everywhere
5. If $f = g$ μ-a.e., then $\int f\,d\mu = \int g\,d\mu$
6. If $f \leq g$ μ-a.e., then $\int f\,d\mu \leq \int g\,d\mu$
7. The map $A \mapsto \int_A \varphi\,d\mu$ is a measure on $\mathcal{M}$.

Proof of 7: Define $\mu_\varphi(A) := \int_A \varphi\,d\mu$. Clearly, $\mu_\varphi(\emptyset) = \int_\emptyset \varphi\,d\mu = \int \varphi \chi_\emptyset\,d\mu = \int 0\,d\mu = 0$.

Now, if $A_1, A_2, \ldots \in \mathcal{M}$ are pairwise disjoint & $A = \bigcup_{i=1}^\infty A_i$, then $\varphi = \sum_{j=1}^n c_j \chi_{E_j}$ and

$$\mu_\varphi(A) = \int_A \varphi\,d\mu = \int \varphi \chi_A\,d\mu = \int \varphi \sum_{i=1}^\infty \chi_{A_i}\,d\mu = \int \sum_{j=1}^n c_j \chi_{E_j} \sum_{i=1}^\infty \chi_{A_i}$$

$$= \sum_{j=1}^n c_j \mu(E_j) \sum_{i=1}^n \chi_{A_i}$$

$$= \sum_{i=1}^\infty \sum_{j=1}^n c_j \chi_{A_i} \mu(E_j)$$

$$= \sum_{i=1}^n \int_{A_i} \varphi\,d\mu$$

$$= \sum_{i=1}^\infty \mu_\varphi(A_i). \qquad \blacksquare$$

Now, of course, we extend the integral to all of L^+ in the obvious way: for $f \in L^+$, define

$$\int f\,d\mu := \sup\left\{ \int \varphi\,d\mu : 0 \leq \varphi \leq f \text{ is simple} \right\}$$