Math 617: Day 10

Measurable Functions & the Lebesgue Integral

As already alluded to, one of the main motivations for defining the Lebesgue measure is to define the Lebesgue integral, which will be more powerful than the Riemann integral, allowing us to, for example, integrate the Dirichlet function $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{else}\end{cases}$

Recall how we defined the Riemann integral $\int_a^b f(x)dx$ of a function $f:[a,b]\to\mathbb{R}$. Usually this is done in terms of Riemann sums, but this is equivalent to setting the integral equal to the lower Darboux integral:

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx := \sup\left\{ p.c.\int_a^b g(x)dx : g\le f,\ g \text{ piecewise constant}\right\}$$

where $p.c.\int_a^b g(x)dx$ is a piecewise constant integral, where we just write the piecewise constant function g as

$$g(x)=\sum_{i=1}^n c_i \chi_{I_i},$$ a linear combination of indicator functions for intervals $I_1,\dots,I_n$, and then

$$p.c.\int_a^b g(x)dx = \sum_{i=1}^n c_i |I_i| .$$

The basic idea of the Lebesgue integral is to do the exact same thing, but where the sets I_i can be arbitrary measurable sets.

Ex: For example, the Dirichlet function $f:[0,1]\to\mathbb{R}$ given by $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{else}\end{cases}$ is already piecewise constant in the Lebesgue sense: $f(x)=\chi_{\mathbb{Q}\cap[0,1]}$, so $\int_0^1 f(x)dx = 1\cdot m(\mathbb{Q}\cap[0,1]) = 0$.

Quote from Terry Tao's lecture notes:

> **Remark 2** The "Lebesgue philosophy" that one is willing to lose control on sets of measure zero is a perspective that distinguishes Lebesgue-type analysis from other types of analysis, most notably that of descriptive set theory, which is also interested in studying subsets of $\mathbf{R}^d$, but can give completely different structural classifications to a pair of sets that agree almost everywhere. This loss of control on null sets is the price one has to pay for gaining access to the powerful tool of the Lebesgue integral; if one needs to control a function at absolutely *every* point, and not just almost every point, then one often needs to use other tools than integration theory (unless one has some regularity on the function, such as continuity, that lets one pass from almost everywhere true statements to everywhere true statements).

Def: Let $(X,\mathcal{M})$ be a measurable space. A function $f:X\to\mathbb{R}$ is measurable if $f^{-1}(E)\in\mathcal{M}$ $\forall E\in\mathcal{B}_{\mathbb{R}}$; similarly for $f:X\to\mathbb{C}$.

More particularly, $f:\mathbb{R}\to\mathbb{R}$ is Borel Measurable if $f^{-1}(E)\in\mathcal{B}_{\mathbb{R}}$ $\forall E\in\mathcal{B}_{\mathbb{R}}$, & f is Lebesgue measurable if $f^{-1}(E)\in\mathcal{L}$ (the Lebesgue measurable sets)

Prop: Suppose $(X,\mathcal{M})$ is a measurable space & $f: X \to \mathbb{R}$. TFAE:

① f is measurable

② $f^{-1}((a,\infty)) \in \mathcal{M}$ $\forall\, a \in \mathbb{R}$ (i.e. $\{x \in X : f(x) > a\} \in \mathcal{M}$)

③ $f^{-1}([a,\infty)) \in \mathcal{M}$ $\forall\, a \in \mathbb{R}$

④ $f^{-1}((-\infty,a)) \in \mathcal{M}$ $\forall\, a \in \mathbb{R}$

⑤ $f^{-1}((-\infty,a]) \in \mathcal{M}$ $\forall\, a \in \mathbb{R}$

Prop: $f: X \to \mathbb{C}$ is measurable $\Leftrightarrow$ $\mathrm{Re}(f)$ & $\mathrm{Im}(f)$ are measurable.

Prop: If $f,g: X \to \mathbb{R}$ (or $\mathbb{C}$) are measurable, then $f+g$ & fg are also measurable.

We can extend to $\overline{\mathbb{R}} = [-\infty,\infty]$, where $\mathcal{B}_{\overline{\mathbb{R}}} = \{E \subseteq \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$.

Prop: If $\{f_n\}$ is a sequence of $\overline{\mathbb{R}}$-measurable functions on $(X,\mathcal{M})$, then

$$g_1(x) = \sup_n f_n(x) \qquad g_3(x) = \limsup_{n\to\infty} f_n(x)$$

$$g_2(x) = \inf_n f_n(x) \qquad g_4(x) = \liminf_{n\to\infty} f_n(x)$$

are all measurable. If $f(x) = \lim_{n\to\infty} f_n(x)$ exists $\forall\, x \in X$, then f is measurable.

Proof: We use the earlier proposition that, e.g., $g_1^{-1}((a,\infty]) \in \mathcal{M} \Rightarrow g_1$ measurable. Specifically,

$$g_1^{-1}((a,\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((a,\infty]) \in \mathcal{M} \text{ since each } f_n^{-1}((a,\infty]) \in \mathcal{M}.$$

Similarly, $g_2^{-1}([-\infty,a)) = \bigcup_{n=1}^{\infty} f_n^{-1}([-\infty,a)) \in \mathcal{M}$, so g_2 is measurable.

Now, we can write $g_3(x) = \inf_k h_k(x)$ where $h_k(x) = \sup_{n \geq k} f_n(x)$. By the above, each h_k is measurable, so g_3 is as well.

Similarly for g_4.

Finally, if $f = \lim_{n\to\infty} f_n$ exists, then $f = g_3 = g_4$ is measurable. ∎

Cor 1: If $f,g: X \to \overline{\mathbb{R}}$ are measurable, then so are $\max(f,g)$ & $\min(f,g)$.

Cor 2: If $\{f_n\}$ is a sequence of complex-valued measurable functions & $f = \lim_{n\to\infty} f_n$ exists, then f is measurable.

For both real- & complex-valued functions, it's handy to have the following decompositions:

- If $f: X \to \overline{\mathbb{R}}$, then the positive part of f is $f^+(x) = \max(f(x), 0)$ & the negative part of f is $f^-(x) = \min(f(x), 0)$.

  By Corollary 1, these are both measurable.

- If $f: X \to \mathbb{C}$, we have the polar decomposition

$$f = |f| \, \mathrm{sgn}(f) \quad \text{where } \mathrm{sgn}(f) = \frac{f}{|f|}$$

(in other words, this is just writing f in $re^{i\theta}$ form)